On the lower semicontinuous quasiconvex envelope for unbounded integrands (II): Representation by generalized controls

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1. Introduction.

a) Nonconvex relaxation of multidimensional control problems.

With the present paper, we continue a series of publications concerned with existence and relaxation theorems for multidimensional control problems of Dieudonné-Rashevsky type:

\( F(x) = \int_{\Omega} f_0(t, x(t), Jx(t)) dt \rightarrow \inf {\text{!}} \quad x \in W^{1,\infty}(\Omega, \mathbb{R}^n) \); (1.1)

\[ Jx(t) = \begin{pmatrix}
\frac{\partial x_1}{\partial t_1}(t) & \cdots & \frac{\partial x_1}{\partial t_m}(t) \\
\vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial t_1}(t) & \cdots & \frac{\partial x_n}{\partial t_m}(t)
\end{pmatrix} \in \mathbb{R}^{n \times m} \quad (\forall) t \in \Omega. \] (1.2)

Here the dimensions are \( n \geq 1, m \geq 2 \), while \( \Omega \subset \mathbb{R}^m \) is the closure of a bounded Lipschitz domain, \( K \subset \mathbb{R}^{nm} \) is a convex body with \( o \in \text{int}(K) \), and \( f_0(t, \xi, v) : \Omega \times \mathbb{R}^n \times K \rightarrow \mathbb{R} \) is a continuous, in general nonconvex integrand. Problems of this kind find applications e. g. in models for the torsion of prismatic bars,\(^{01}\) in optimization problems for convex bodies under geometrical restrictions\(^{02}\) and within the framework of image processing.\(^{03}\) In their papers on underdetermined boundary value problems for nonlinear first-order PDE’s,\(^{04}\) DACOROGNA and MARCELLINI arrived at Dieudonné-Rashevsky type problems as well.

b) The lower semicontinuous quasiconvex envelope.

In order to extend the known relaxation results in multidimensional control to the vectorial case \( (n \geq 2) \),\(^{05}\) the author introduced an appropriate quasiconvex envelope for unbounded integrands \( f : \mathbb{R}^{nm} \rightarrow \mathbb{R} = \mathbb{R} \cup \{ (+\infty) \} \). For such functions, the notion of quasiconvexity must be precised in the following way:

**Definition 1.1. (Quasiconvex function with values in \( \mathbb{R} \))** A function \( f : \mathbb{R}^{nm} \rightarrow \mathbb{R} \) with the following properties is said to be quasiconvex:

1) \( \text{dom}(f) \subseteq \mathbb{R}^{nm} \) is a (nonempty) Borel set;
2) \( f \big|_{\text{dom}(f)} \) is Borel measurable and bounded from below on every bounded subset of \( \text{dom}(f) \);
3) for all \( v \in \mathbb{R}^{nm} \), \( f \) satisfies Morrey’s integral inequality:

\[ f(v) \leq \frac{1}{|\Omega|} \int_{\Omega} f(v + Jx(t)) dt \quad \forall x \in W^{1,\infty}(\Omega, \mathbb{R}^n); \] (1.3)

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01) [Lur'e 75], pp. 240 ff., [Ting 69a], p. 531 f., [Ting 69b], [Wagner 96], pp. 76 ff.
02) [Andrejewa/Klötzler 84a], [Andrejewa/Klötzler 84b], p. 149 f.
03) [Brune 07], [Franek 07a], [Franek 07b], [Wagner 07a].
04) [DACOROGNA/MARCELLINI 97], [DACOROGNA/MARCELLINI 98], [DACOROGNA/MARCELLINI 99].
05) Cf. [Wagner 07c].
06) [Wagner 07a], p. 6, Definition 2.9., as specification of [Ball/Murat 84], p. 228, Definition 2.1., in the case \( p = (+\infty) \).
or equivalently
\[ f(v) = \inf \left\{ \frac{1}{|\Omega|} \int_{\Omega} f(v + Jx(t)) \, dt \mid x \in W^{1,\infty}_0(\Omega, \mathbb{R}^n) \right\}. \]  \hfill (1.4)

Here \( \Omega \subset \mathbb{R}^m \) is the closure of a bounded strongly Lipschitz domain.

The lower semicontinuous quasiconvex envelope of an unbounded function is then defined as follows:

**Definition 1.2.** (Lower semicontinuous quasiconvex envelope \( f^{qc} \) for functions with values in \( \mathbb{R}^m \)) To any function \( f : \mathbb{R}^{nm} \to \mathbb{R} \) bounded from below, we define
\[
f^{qc}(v) = \sup \left\{ g(v) \mid g : \mathbb{R}^{nm} \to \mathbb{R} \text{ quasiconvex and lower semicontinuous,} \quad g(v) \leq f(v) \quad \forall v \in \mathbb{R}^{nm} \right\}. \tag{1.5}
\]

Obviously, Definition 1.2. generalizes the usual formation of a quasiconvex envelope since the quasiconvex functions \( g \) below a finite function \( f \) are continuous from the outset. Assume now that \( K \subset \mathbb{R}^{nm} \) is a convex body with \( \emptyset \in \text{int}(K) \) and \( f : \mathbb{R}^{nm} \to \mathbb{R} \) is a function with \( f \mid K \in C^0(K, \mathbb{R}) \) and \( f \mid (\mathbb{R}^{nm} \setminus K) \equiv (+\infty) \). In this situation, the author proved that \( f^{qc} \) may be represented in terms of Jacobi matrices in the following way:

**Theorem 1.3.** (Representation of \( f^{qc} \) in terms of Jacobi matrices) Under the assumptions mentioned above, the lower semicontinuous quasiconvex envelope \( f^{qc} : \mathbb{R}^{nm} \to \mathbb{R} \) admits the representation
\[
f^{qc}(v_0) = \begin{cases} 
  \lim_{v \to v_0, v \in R \cap \text{int}(K)} f^*(v) & \mid v_0 \in \text{int}(K); \\
  f^*(v) & \mid v_0 \in \partial K; \\
  (+\infty) & \mid v_0 \in R^{nm} \setminus K,
\end{cases} \tag{1.6}
\]

where \( R = \overrightarrow{\sigma v_0} \) denotes the ray through \( v_0 \) starting from the origin, and \( f^*(v_0) \) is defined by
\[
f^*(v_0) = \inf \left\{ \frac{1}{|\Omega|} \int_{\Omega} f(v_0 + Jx(t)) \, dt \mid x \in W^{1,\infty}_0(\Omega, \mathbb{R}^n), v_0 + Jx(t) \in K \quad \forall t \in \Omega \right\} \in \mathbb{R}. \tag{1.7}
\]

c) **Representation of \( f^{qc} \) by probability measures.**

The present paper is concerned with the question how to represent \( f^{qc} \) in terms of probability measures. In the situation of Theorem 1.3., it is well-known that for all \( w \in K \), the convex envelope of \( f \) admits the description \( f^c(w) = \text{Min} \left\{ \int_K f(v) \, d\nu(v) \mid \nu \in S^c(w) \right\} \)
\[
\text{where } S^c(w) = \left\{ \nu \in rca(\mathbb{R}^{nm}) \mid \nu \text{ is a probability measure,} \quad \text{supp}(\nu) \subseteq K, \quad w = \left( \begin{array}{c}
 f_K v_{11} \, d\nu(v) \\
 \vdots \\
 f_K v_{nm} \, d\nu(v)
\end{array} \right) \right\}. \tag{1.9}
\]

(07) [Wagner 07a], p. 9, Definition 2.14., 2).
(08) [Dacorogna 89], p. 29, Theorem 2.3., 2).
(09) [Wagner 07a], p. 29, Theorem 4.1.
(10) Cf. [Wagner 06a], p. 131, Theorem 10.19.
In the following, we search for an analogous description of \( f^{(qc)} \), depending on subsets \( S^{(qc)}(w) \subseteq S^c(w) \).\(^{11}\) For this purpose, we adopt the averaging technique for Young measures introduced by Kinderlehrer and Pedregal.\(^{12}\) Analogously to the approach pursued in [Wagner 07a], the desired set-valued map \( S^{(qc)} \) will be constructed then in two steps: we define first a map \( S^* \) on \( \text{int}(K) \) and then continue it to \( \partial K \) by a radial limit passage.\(^{13}\) As the main result, we obtain the following representation theorem for \( f^{(qc)} \):

**Theorem 1.4. (Representation of \( f^{(qc)} \) in terms of probability measures)** Assume that \( K \subset \mathbb{R}^{nm} \) is a convex body with \( a \in \text{int}(K) \) and \( f : \mathbb{R}^{nm} \to \mathbb{R} \) is a function with \( f | K \in C^0(K, \mathbb{R}) \) and \( f | (\mathbb{R}^{nm} \setminus K) \equiv (+\infty) \). Then for all \( w \in K \), \( f^{(qc)}(w) \) may be represented as

\[
f^{(qc)}(w) = \text{Min} \left\{ \int_K f(v) \, d\nu(v) \mid \nu \in S^{(qc)}(w) \right\}.
\] (1.10)

Here \( S^{(qc)}(w) \) is a set-valued map with nonempty, convex, weak*-sequentially compact images, which will be described in Definitions 3.1. and 3.8. below.

d) Outline of the paper.

In Section 2, we consider first the metric space \( rca^{pr}(K) \) of the probability measures supported on \( K \), then the set \( \mathcal{Y}(K) \) of generalized controls (“Young measures”) and the subset \( \mathcal{G}(K) \) of those generalized controls, which can be generated by Jacobi matrices (generalized gradient controls, “gradient Young measures”). Then we provide an appropriate version of Kinderlehrer/Pedregal’s mean value theorem, accompanied by a complete proof. In Section 3, we define the desired map \( S^{(qc)} \) in two steps: at first as a continuous set-valued map \( S^* \) on \( \text{int}(K) \), and then as the upper semicontinuous extension \( S^# \) of \( S^* \) to \( \partial K \). It turns out that the \( \varepsilon-\delta \) relations for the functions \( f^* \) and \( f^# \) from [Wagner 07a]\(^{14}\) may be rephrased within the context of set-valued maps where the upper semicontinuity of the set-valued map \( S^# \) takes the place of the lower semicontinuity of the function \( f^# = f^{(qc)} \). Section 4 is devoted to the proof of Theorem 1.4. Finally, in the Appendix the concepts from the theory of set-valued maps (Painlevé-Kuratowski limits, semicontinuity and continuity) have been summarized.

e) Notations and abbreviations.

Let \( k \in \{0, 1, \ldots, \infty\} \) and \( 1 \leq p \leq \infty \). Then \( C^k(\Omega, \mathbb{R}^r) \), \( L^p(\Omega, \mathbb{R}^r) \) and \( W^{k,p}(\Omega, \mathbb{R}^r) \) denote the spaces of \( r \)-dimensional vector functions whose components are \( k \)-times continuously differentiable, belong to \( L^p(\Omega) \) or to the Sobolev space of \( L^p(\Omega) \)-functions with weak derivatives up to \( k \)th order in \( L^p(\Omega) \) respectively. In addition, functions within the subspaces \( C^k_0(\Omega, \mathbb{R}^r) \subset C^k(\Omega, \mathbb{R}^r) \) are compactly supported while functions within the subspace \( W^{1,\infty}_0(\Omega, \mathbb{R}^r) \subset W^{1,\infty}(\Omega, \mathbb{R}^r) \) admit a (Lipschitz-) continuous representative\(^{15}\) with zero boundary values. The symbol \( \partial x/\partial t_j \) may denote the classical as well as the weak partial derivative of \( x \) by \( t_j \). The Jacobi matrix of \( x \) is abbreviated as \( Jx \).

The space of Radon measures (signed regular measures) acting on the \( \sigma \)-algebra of the Borel subsets of a compact set \( K \subset \mathbb{R}^{nm} \) is denoted by \( rca(K) \). Endowed with the total variation norm \( \gamma \mu(K) \), it forms a

\(^{11}\) See [Pedregal 97], p. 8 ff., Section 1.3.

\(^{12}\) [Kinderlehrer/Pedregal 91], [Pedregal 97].

\(^{13}\) The limit has to be understood in the sense of Painlevé/Kuratowski, see the Appendix.

\(^{14}\) See [Wagner 07a], p. 16, Theorem 3.5., p. 22 f., Theorem 3.12., and p. 23, Theorem 3.15.

\(^{15}\) [Evans/Gariepy 92], p. 131, Theorem 5.
Banach space.\(^{16}\) Due to the compactness of \(K\), the dual space \((C^0(K, \mathbb{R}))^*\) and \(\text{rea} (K)\) are isomorphical,\(^ {17}\) consequently, every linear, continuous functional on \(C^0(K, \mathbb{R})\) may be represented as an integral with respect to a Radon measure \(\nu \in \text{rea} (K)\). The subset of the probability measures, equipped with a suitable metric, will be denoted by \(\text{rea}^{P \nu} (K)\) (see Definition 2.3. below). The Dirac measure concentrated in \(v \in K\) is denoted by \(\delta_v\).

We denote by \(\text{int} (A)\), \(\partial A\), \(\text{cl} (A)\), and \(| A |\) the interior, boundary, closure, the convex hull and the \(r\)-dimensional Lebesgue measure of the set \(A \subseteq \mathbb{R}^r\), respectively. \(\mathbb{I}_A : \mathbb{R}^r \to \mathbb{R}\) with \(\mathbb{I}_A (t) = 1 \iff t \in A\) and \(\mathbb{I}_A (t) = 0 \iff t \notin A\) is the characteristic function of the set \(A \subseteq \mathbb{R}^r\). We set \(\mathbb{R} = \mathbb{R} \cup \{ (+\infty) \}\) and equip \(\mathbb{R}\) with the natural topological and order structures where \((+\infty)\) is the greatest element. Throughout the whole paper, we consider only proper functions \(f : \mathbb{R}^m \to \mathbb{R}\), assuming that \(\text{dom} (f) = \{ v \in \mathbb{R}^m \mid f(v) < (+\infty) \}\) is always nonempty. The restriction of the function \(f\) to the subset \(A\) of its range of definition is denoted by \(f \upharpoonright A\).

**Definition 1.5. (Function class \(\mathcal{F}_K\))** Let \(K \subseteq \mathbb{R}^m\) be a given convex body with \(\sigma \in \text{int} (K)\). We say that a function \(f : \mathbb{R}^m \to \mathbb{R}\) belongs to the class \(\mathcal{F}_K\) iff \(f \mid K \in C^0 (K, \mathbb{R})\) and \(f \big| (\mathbb{R}^m \setminus K) \equiv (+\infty)\).

Consequently, any function \(f \in \mathcal{F}_K\) is bounded and uniformly continuous on \(K\), and the class \(\mathcal{F}_K\) and the Banach space \(C^0 (K, \mathbb{R})\) are isomorphical and isometrical.

If \(X\) is an arbitrary set then \(\mathcal{B}(X)\) denotes the set of all subsets of \(X\). For the definition of the Painlevé-Kuratowski limits \(\liminf_{K \to \infty} E_N, \limsup_{K \to \infty} E_N\) and \(\lim^K_{N \to \infty} E_N\) of a set sequence \(\{ E_N \}\), we refer to the Appendix.

We close this subsection with three nonstandard notations. \("\{ x^N \}, \Lambda\) denotes a sequence \(\{ x^N \}\) with members \(x^N \in A\). If \(A \subseteq \mathbb{R}^r\) then the abbreviation \("(\forall) t \in \Lambda\)" has to be read as "for almost all \(t \in \Lambda\)" resp. "for all \(t \in A\) except a \(r\)-dimensional Lebesgue null set". The symbol \(\sigma\) denotes, depending on the context, the zero element resp. the zero function of the underlying space.

**2. Generalized controls.**

**a) The metric space \(\text{rea}^{P \nu} (K)\) of the probability measures supported on \(K\).**

Denote by \(X\) the (norm-)closed unit ball of the Banach space \(\text{rea} (K)\). We introduce the following metric on \(X\):

**Definition 2.1. (The function \(\sigma\))** Assume that countably many functions \(g_s \in C^0 (K, \mathbb{R}) \cap W^{1, \infty} (K, \mathbb{R})\) with \(\| g_s \|_{C^0 (K, \mathbb{R})} = 1\) and Lipschitz constants \(L_s > 0\) form a dense subset \(\{ g_s \}\) of the unit ball of \(C^0 (K, \mathbb{R})\) with respect to its norm topology. The we define a function \(\sigma : X \times X \to \mathbb{R}\) by

\[
\sigma (\nu', \nu'') = \sum_{s=1}^{\infty} \frac{1}{2^{1+s} (1 + L_s)} \left| \int_K g_s (v) \left( d\nu' (v) - d\nu'' (v) \right) \right| .
\]

(2.1)

The next lemma shows that \(\sigma\) is a distance function on \(X\). The topology generated by \(\sigma\) and the weak* topology on \(X\) coincide.

**Lemma 2.2. (\(\sigma\) as a distance function on \(X\))** Given a countable, dense subset \(\{ g_s \}, C^0 (K, \mathbb{R})\) of the closed unit ball of \(C^0 (K, \mathbb{R})\) and a convergent series \(\sum_{s=1}^{\infty} a_s\) with positive members \(a_s > 0\). Then

\[
\sigma (\nu', \nu'') = \sum_{s=1}^{\infty} a_s \left| \int_K g_s (v) \left( d\nu' (v) - d\nu'' (v) \right) \right| .
\]

(2.2)

\(^{16}\) [Dunford/Schwartz 88], p. 161 f.

\(^{17}\) Ibid., p. 265, Theorem 3.
is a distance function with
\[ \{ \nu^N \} , X \xrightarrow{\ast} \nu \iff \sigma(\nu^N, \nu) \to 0. \] (2.3)

By Alaoglu’s theorem,\(^{18}\) together with the metric \(\sigma\) from Definition 2.1., \(X\) forms a compact metric space. The probability measures \(\nu\) with \(\text{supp}(\nu) \subseteq K\) form a convex, weak*-closed subset of \(X\)\(^{19}\) and, consequently, a compact metric subspace of \([X, \sigma]\). This motivates the following definition:

**Definition 2.3. (Metric space of the probability measures on \(K\))** The subset of the probability measures \(\nu \in \text{rca}^* (K)\), endowed with the metric \(\sigma\) from Definition 2.1., will be denoted by \(\text{rca}^* (K)\).

**Theorem 2.4. (Dense subsets of \(\text{rca}^* (K)\))**

1)\(^{20}\) The subset of all finite convex combinations of Dirac measures,
\[ \{ \sum_{k=1}^{K} \lambda_k \delta_{v_k} \mid \sum_{k=1}^{K} \lambda_k = 1 , \ 0 \leq \lambda_k \leq 1 , \ v_k \in K , \ 1 \leq k \leq K , \ K \in \mathbb{N} \} , \] (2.4)
is dense in \(\text{rca}^* (K)\).

2) The countable subset
\[ \{ \sum_{k=1}^{K} \lambda_k \delta_{v_k} \mid \sum_{k=1}^{K} \lambda_k = 1 , \ \lambda_k \in [0, 1] \cap Q , \ v_k \in K \cap Q^{sm} , \ 1 \leq k \leq K , \ K \in \mathbb{N} \} \] (2.5)
is dense in \(\text{rca}^* (K)\) as well; consequently, the metric space \(\text{rca}^* (K)\) is separable.

**Proof of Lemma 2.2.** Two linear, continuous functionals \(\nu', \nu'' \in \text{rca} (K)\) are identical iff \(\langle g_s , \nu' - \nu'' \rangle = 0\) for all \(s \in \mathbb{N}\). The symmetry \(\sigma(\nu', \nu'') = \sigma(\nu'', \nu')\) is obvious as well. The triangle inequality results from
\[ \sigma(\nu', \nu'') = \sum_{a=1}^{\infty} a_s \left| \int_K g_s(v) \left( dv'(v) - dv''(v) \right) \right| \leq \sum_{a=1}^{\infty} a_s \left( \left| \int_K g_s(v) \left( dv'(v) - dv''(v) \right) \right| + \left| \int_K g_s(v) \left( dv''(v) - dv'(v) \right) \right| \right) . \] (2.6)
If a sequence with \(\{ \nu^N \} , X \xrightarrow{\ast} \nu \in X\) is given then the norm differences \(\| \nu^N - \nu \| \leq 2\) are bounded. We fix \(\varepsilon > 0\). Since the series \(\sum_{s=1}^{\infty} a_s\) is convergent, we find some index \(s_0 \in \mathbb{N}\) with
\[ \sum_{s=s_0+1}^{\infty} a_s \cdot \| g_s \| \cdot \| \nu^N - \nu \| \leq 2 \cdot \sum_{s=s_0+1}^{\infty} a_s \leq \frac{\varepsilon}{2} \] (2.7)
for all \(N \in \mathbb{N}\). Further, we may choose \(N(\varepsilon) \in \mathbb{N}\) such that the estimate
\[ \sum_{s=1}^{s_0} a_s \left| \int_K g_s(v) \left( dv'^N (v) - dv(v) \right) \right| \leq \frac{\varepsilon}{2} \] (2.8)
holds for all \(N \geq N(\varepsilon)\). Both estimates together show that for all \(\varepsilon > 0\) there exists \(N(\varepsilon) \in \mathbb{N}\) with
\[ 0 \leq \sigma(\nu^N, \nu) \leq \varepsilon \ \forall N \leq N(\varepsilon) ; \] (2.9)

\(^{18}\) [DUNFORD/SCHWARTZ 88], p. 424, Theorem 2.

\(^{19}\) [ROUBÍČEK 97], p. 47 f., Proposition 1.5.1. (iii).

\(^{20}\) Ibid., p. 47 f., Proposition 1.5.1. (iv).
consequently, we find that \( \lim_{N \to \infty} \sigma(\nu^N, \nu) = 0 \). Conversely, consider a sequence \( \{ \nu^N \} \), \( X \) with \( \sigma(\nu^N, \nu) \to 0 \). Fix some index \( s_0 \in \mathbb{N} \) and choose \( \varepsilon > 0 \). Then there exists \( N(\varepsilon, s_0) \in \mathbb{N} \) with \( \sigma(\nu^N, \nu) \leq a_{s_0} \varepsilon \) resp. 

\[
\sum_{s=1}^{\infty} a_s \left| \int_K g_s(v) \left( d\nu^N(v) - d\nu(v) \right) \right| = \sum_{s=1, s \neq s_0}^{\infty} a_s \left| \int_K g_s(v) \left( d\nu^N(v) - d\nu(v) \right) \right| + a_{s_0} \left| \int_K g_{s_0}(v) \left( d\nu^N(v) - d\nu(v) \right) \right| \leq a_{s_0} \varepsilon
\]

for all \( N \geq N(\varepsilon, s_0) \). Since both members are nonnegative, we may particularly conclude that

\[
a_{s_0} \left| \int_K g_{s_0}(v) \left( d\nu^N(v) - d\nu(v) \right) \right| \leq a_{s_0} \varepsilon \quad \forall \ N \geq N(\varepsilon, s_0)
\]

and

\[
\lim_{N \to \infty} \left( g_{s_0}, \nu^N - \nu \right) = 0
\]

for all \( s_0 \in \mathbb{N} \), from which the weak*-convergence \( \{ \nu^N \} \), \( X \xrightarrow{w^*} \nu \) follows. 

**Proof of Theorem 2.4. 2)** By Part 1), we find for arbitrary \( \nu \in \text{rca}^p(K) \) and \( \varepsilon > 0 \) a convex combination of Dirac measures with \( \sigma(\nu, \sum_{k=1}^{K} \lambda_k \delta_{\nu_k}) \leq \varepsilon \). Then we may choose numbers \( \lambda_k^e \in [0, 1] \cap \mathbb{Q} \) as well as points \( v_k^e \in K \cap \mathbb{Q}^{1m} \) with \( |\lambda_k - \lambda_k^e| \leq \varepsilon/K \) and \( |v_k - v_k^e| \leq \varepsilon \), \( 1 \leq k \leq K \) as well as \( \sum_{k=1}^{K} \lambda_k^e = 1 \). Then it holds:

\[
\sigma \left( \sum_{k=1}^{K} \lambda_k \delta_{v_k}, \sum_{k=1}^{K} \lambda_k^e \delta_{v_k^e} \right) = \sum_{s=1}^{\infty} 2^{1+s} \left( \frac{1}{1 + L_s} \right) | \sum_{k=1}^{K} \int_{K} g_s(v) \left( \lambda_k \delta_{v_k} - \lambda_k^e \delta_{v_k^e} \left( v_k \right) \right) |
\]

\[
\leq \sum_{s=1}^{\infty} 2^{1+s} \left( \frac{1}{1 + L_s} \right) \left( | \lambda_k \delta_{v_k} - \lambda_k^e \delta_{v_k^e} \left( v_k \right) | + | \sum_{k=1}^{K} \int_{K} g_s(v) \lambda_k^e (\delta_{v_k} - \delta_{v_k^e}(v)) | \right)
\]

\[
\leq \sum_{s=1}^{\infty} 2^{1+s} \left( \frac{1}{1 + L_s} \right) \sum_{k=1}^{K} \left( | g_s(v_k^e) | \cdot | \lambda_k - \lambda_k^e | + \lambda_k^e \cdot | g_s(v_k) - g_s(v_k^e) | \right)
\]

\[
\leq \sum_{s=1}^{\infty} 2^{1+s} \left( \frac{1}{1 + L_s} \right) \sum_{k=1}^{K} \left( \| g_s \|_{C^0(K)} \cdot \frac{\varepsilon}{K} + \lambda_k^e \cdot L_s \cdot | v_k - v_k^e | \right)
\]

Consequently, with respect to the metric \( \sigma, \nu \) may be approximated arbitrary closely by elements of the subset defined in Part 2) of the Theorem.

**b) Generalized controls (“Young measures”).**

We start with the definition of the set \( \mathcal{Y}(K) \) of generalized controls for (P). Throughout this subsection, we assume that \( \Omega \subset \mathbb{R}^m \) is the closure of a strongly Lipschitz domain.

**Definition 2.5. (Generalized controls, “Young measures”)** A measure-valued map \( \mu : \Omega \to \text{rca}^p(K) \) with \( t \mapsto \mu_t \) is called a generalized control if, for any continuous function \( g \in C^0(K, \mathbb{R}) \), the function \( h_g(t) = \int_K g(v) d\mu_t(v) \) is Borel measurable on \( \Omega \). Two generalized controls \( \mu^r = \{ \mu^r_t \} \) and \( \mu^l = \{ \mu^l_t \} \) will be identified if \( \mu^r_t \equiv \mu^l_t \) holds for almost all \( t \in \Omega \). The set of all equivalence classes of generalized controls will be denoted by \( \mathcal{Y}(K) \).

**Remarks.** a) In the literature, the elements of \( \mathcal{Y}(K) \) are commonly called “Young measures” or “parametrized measures”. \(^{21}\) We prefer, however, the use of the term “generalized control” introduced by GAM-

\(^{21}\) [Kinderlehrer/Pedregal 91], p. 331 ff., [Müller 99], p. 115 ff., [Pedregal 97], p. 20 ff., etc.
KREIDZE,\textsuperscript{22} since we must carefully distinguish between (equivalence classes of) measure-valued maps and single measures, resulting from these maps by an averaging process.

b) The relation between generalized and “ordinary” controls for (P) will be established in Theorem 2.11.

Equipped with the topology introduced below, \( \mathcal{Y}(K) \) becomes a sequentially compact topological space:

**Definition 2.6. (Topology on \( \mathcal{Y}(K) \))**\textsuperscript{23} The convergence of a sequence \( \{ \mu^N \} \), \( \mathcal{Y}(K) \) towards the limit \( \mu \in \mathcal{Y}(K) \) is defined through

\[
\mu^N \to \mu \iff \int_{\Omega} \int_K f(t) g(v) (d\mu^N_t(v) - d\mu_t(v)) \, dt \to 0 \quad \text{for all } f \in L^1(\Omega, \mathbb{R}), \ g \in C^0(K, \mathbb{R}).
\]  

(2.18)

**Theorem 2.7. (Compactness of the topological space \( \mathcal{Y}(K) \))**\textsuperscript{24} With respect to the topology from Definition 2.6., the set \( \mathcal{Y}(K) \) is sequentially compact.

Since both spaces \( L^1(\Omega, \mathbb{R}) \) and \( C^0(K, \mathbb{R}) \) are separable, the topology from Definition 2.6. is metrizable.\textsuperscript{25}

**Definition 2.8. (The function \( \varrho \))** Assume that countably many functions \( f_1 \equiv 1/|\Omega|, \ f_r \in C^0(\Omega, \mathbb{R}) \cap L^1(\Omega, \mathbb{R}) \) with \( \| f_r \|_{C^0(\Omega, \mathbb{R})} \cdot |\Omega| = 1 \) for \( r \geq 2 \) as well as \( g_r \in C^0(K, \mathbb{R}) \cap W^{1,\infty}(K, \mathbb{R}) \) with \( \| g_r \|_{C^0(K, \mathbb{R})} = 1 \) and Lipschitz constants \( L_s > 0 \) form dense subsets \( \{ f_r \} \) resp. \( \{ g_r \} \) of the unit balls of \( L^1(\Omega, \mathbb{R}) \) resp. \( C^0(K, \mathbb{R}) \) with respect to their norm topologies. Then we define a function \( \varrho : \mathcal{Y}(K) \times \mathcal{Y}(K) \to \mathbb{R} \) by

\[
\varrho(\mu', \mu'') = \sum_{s=1}^{\infty} \frac{1}{2^{1+s}(1 + L_s)} \left| \int_{\Omega} \int_K g_s(v) (d\mu'_t(v) - d\mu''_t(v)) \, dt \right|
\]

\[
+ \sum_{r=2}^\infty \sum_{s=1}^\infty \frac{1}{2^{r+s}(1 + L_s)} \left| \int_{\Omega} \int_K f_r(t) g_s(v) (d\mu'_t(v) - d\mu''_t(v)) \, dt \right|.
\]  

(2.19)

The following lemma shows that \( \varrho \) is a distance function on \( \mathcal{Y}(K) \), and that the topology generated by \( \varrho \) coincides with the topology introduced in Definition 2.6.

**Lemma 2.9. (\( \varrho \) as a distance function on \( \mathcal{Y}(K) \))** Given a countable, dense subset \( \{ f_r \} \) of the closed unit ball of \( L^1(\Omega, \mathbb{R}) \) and a countable, dense subset \( \{ g_s \} \) of the closed unit ball of \( C^0(K, \mathbb{R}) \). Assume further that the convergent double series \( \sum_{r=1}^\infty \sum_{s=1}^\infty a_{rs} \) with positive members \( a_{rs} > 0 \) satisfies the assumptions of the major rearrangement theorem.\textsuperscript{26} Then

\[
\varrho(\mu', \mu'') = \sum_{r=1}^\infty \sum_{s=1}^\infty a_{rs} \left| \int_{\Omega} \int_K f_r(t) g_s(v) (d\mu'_t(v) - d\mu''_t(v)) \, dt \right|
\]  

(2.20)

is a distance function on \( \mathcal{Y}(K) \) with \( \{ \mu^N \} \), \( \mathcal{Y}(K) \to \mu \iff \varrho(\mu^N, \mu) \to 0. \)

By means of the metric \( \varrho \), we introduce the notion of a generating function sequence for a generalized control \( \mu \).

\textsuperscript{22} [Gamkrelidze 78], p. 23.
\textsuperscript{23} Cf. [Berliocchi/Lasry 73], p. 141, and [Gamkrelidze 78], p. 29.
\textsuperscript{24} [Berliocchi/Lasry 73], p. 144, Proposition 1, (i); independently proved again in [Kraut/Pickenhain 90], p. 391, Theorem 4.
\textsuperscript{25} Cf. [Kinderlehrer/Pedregal 91], p. 337.
\textsuperscript{26} [Knopp 64], p. 144.
Definition 2.10. (Generating sequences for generalized controls)\(^{27}\) We say that the sequence \(\{ u^N \} \), \(L^\infty(\Omega, \mathbb{R}^{nm})\) generates the generalized control \(\mu \in \mathcal{Y}(K)\) if \(u^N(t) \in K (\forall) t \in \Omega \forall N \in \mathbb{N}\) and

\[
\lim_{N \to \infty} g(\{ \delta_{u^N(t)} \}, \mu) = 0, \quad \text{i. e.}
\]

\[
\lim_{N \to \infty} \int_{\Omega} f(t) g(u^N(t)) \, dt = \lim_{N \to \infty} \int_{K} \int_{\Omega} f(t) g(v) \, d\mu_N(t)(v) \, dt = \int_{K} \int_{\Omega} f(t) g(v) \, d\mu(t)(v) \, dt,
\]

for all \(f \in L^1(\Omega, \mathbb{R}), g \in C_0(K, \mathbb{R})\).

Theorem 2.11. 1) (Existence of a generating subsequence within bounded sequences \(\{ u^N \}, L^\infty(\Omega, \mathbb{R}^{nm})\))\(^{28}\) Every sequence \(\{ u^N \}, L^\infty(\Omega, \mathbb{R}^{nm})\) with \(u^N(t) \in K (\forall) t \in \Omega \forall N \in \mathbb{N}\) admits a weak\(^*\)-convergent subsequence, which generates a generalized control \(\mu \in \mathcal{Y}(K)\).

2) (Generalized controls \(\mu = \{ \delta_{u(t)} \}\) are dense)\(^{29}\) Conversely, the generalized controls of the shape \(\mu = \{ \delta_{u(t)} \}\) with \(u \in L^\infty(\Omega, \mathbb{R}^{nm})\), \(u(t) \in K (\forall) t \in \Omega\), are dense in \(\mathcal{Y}(K)\) with respect to the topology introduced above.

This assertion establishes the relation between “ordinary” and generalized controls. If the control domain \(U = \{ u \in L^\infty(\Omega, \mathbb{R}^{nm}) \mid u(t) \in K (\forall) t \in \Omega\}\) of (P) is embedded into \(\mathcal{Y}(K)\) via \(u \longrightarrow \{ \delta_{u(t)} \}\) then the image of \(U\) forms a dense subset of the sequentially compact space \(\mathcal{Y}(K)\).

Lemma 2.12. (Diagonalization of generating sequences) Consider a sequence \(\{ \mu^M \}, \mathcal{Y}(K) \to \mu \in \mathcal{Y}(K)\) together with generating sequences \(\{ u^{M,N} \}, L^\infty(\Omega, \mathbb{R}^{nm})\) for every \(\mu^M\). Then \(\mu\) is generated by a diagonal sequence \(\{ u^{M,N(M)} \}, L^\infty(\Omega, \mathbb{R}^{nm})\).

Proof of Lemma 2.9. Obviously, it holds that \(g(\mu', \mu'') = 0 \iff \mu'_i \equiv \mu''_i (\forall) t \in \Omega\) and \(g(\mu', \mu'') = g(\mu'', \mu')\). The triangle inequality follows from

\[
g(\mu', \mu'') \leq \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} a_{rs} \left( \left| \int_{K} \int_{\Omega} f_r(t) g_s(v) \left( d\mu'_t(v) - d\mu''_t(v) \right) \, dt \right| + \left| \int_{K} \int_{\Omega} f_r(t) g_s(v) \left( d\mu''_t(v) - d\mu''_t(v) \right) \, dt \right| \right).
\]

Consider now a sequence \(\{ u^N \}, \mathcal{Y}(K) \to \mu \in \mathcal{Y}(K)\). Since all \(\mu^N_t\) and \(\mu_t\) are probability measures, we get the estimate

\[
\left| \int_{K} \int_{\Omega} f_r(t) g_s(v) \left( d\mu'_t(v) - d\mu''_t(v) \right) \, dt \right| \leq \int_{\Omega} \| f_r(t) \| \| g_s \| \| \mu^N_t - \mu_t \| \, dt \leq 2 \quad \forall N \in \mathbb{N}.
\]

Choose a number \(\varepsilon > 0\). Then our assumptions about the double series guarantee the existence of indices \(r_0(\varepsilon), s_0(\varepsilon) \in \mathbb{N}\) with

\[
\sum_{r=r_0+1}^{\infty} \sum_{s=s_0+1}^{\infty} a_{rs} \left| \int_{K} \int_{\Omega} f_r(t) g_s(v) \left( d\mu'_t(v) - d\mu''_t(v) \right) \, dt \right| \leq 2 \sum_{r=r_0+1}^{\infty} \sum_{s=s_0+1}^{\infty} a_{rs} \leq \frac{\varepsilon}{2}
\]

for all \(N \in \mathbb{N}\). Further, we may choose \(N(\varepsilon) \in \mathbb{N}\) such that the estimate

\[
\sum_{r=r_0+1}^{N(\varepsilon)} \sum_{s=s_0+1}^{\infty} a_{rs} \left| \int_{K} \int_{\Omega} f_r(t) g_s(v) \left( d\mu'_t(v) - d\mu''_t(v) \right) \, dt \right| \leq \frac{\varepsilon}{2}
\]

\(^{27}\) Cf. [Pedregal 97], pp. 96 ff.

\(^{28}\) [Müller 99], p. 115 f., Theorem 3.1.

\(^{29}\) [Berliocchi/Lasry 73], p. 148, Proposition 4.
holds for all $N \geq N(\varepsilon)$. Both estimates together show that for all $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ with $0 \leq g(\mu^N, \mu) \leq \varepsilon$ for all $N \geq N(\varepsilon)$; consequently, it holds
\[
\lim_{N \to \infty} g(\mu^N, \mu) = 0. \tag{2.27}
\]
Conversely, let us assume that $g(\mu^N, \mu) \to 0$. Then for arbitrary fixed indices $r_0$, $s_0 \in \mathbb{N}$, we may find to $\varepsilon > 0$ an index $N(\varepsilon, r_0, s_0) \in \mathbb{N}$ with $g(\mu^N, \mu) \leq a_{r_0, s_0} \varepsilon$, i.e.
\[
\sum_{r=1, r \neq r_0} \sum_{s=1, s \neq s_0} a_{rs} \cdot | \ldots | + \sum_{s=1, s \neq s_0} a_{r_0, s} \cdot | \ldots | + a_{r_0, s_0} \int_{K} \int_{\Omega} f_{r_0}(t) g_{s_0}(v) \left( d\mu^N_t(v) - d\mu_t(v) \right) dt \leq a_{r_0, s_0} \varepsilon. \tag{2.28}
\]
Since all members are nonnegative, we derive from (2.28) the inequality
\[
\left| \int_{K} \int_{\Omega} f_{r_0}(t) g_{s_0}(v) \left( d\mu^N_t(v) - d\mu_t(v) \right) dt \right| \leq \varepsilon. \tag{2.29}
\]
Consequently, the condition from Definition 2.6. is satisfied for all functions $f_{r_0}$, $g_{s_0}$ from dense subsets of the unit balls of $L^1(\Omega, R)$ resp. $C^0(K, R)$. The desired relation $\mu^N \to \mu$ follows.

**Proof of Lemma 2.12.** Our assumptions read as
\[
\lim_{M \to \infty} g(\mu^M, \mu) = 0 \quad \text{and} \quad \lim_{N \to \infty} g(\{ \delta_{u^{M,N}(t)} \}, \mu^M) = 0 \tag{2.30}
\]
for all $M \in \mathbb{N}$. By a passage to subsequences, we may rearrange the indices in such a way that $\phi(\mu^M, \mu) = 1/M$ as well as $\phi(\{ \delta_{u^{M,N}(t)} \}, \mu^M) = 1/N$ hold for all $M$, $N \in \mathbb{N}$. It follows that $\phi(\{ \delta_{u^{M,M,N}(t)} \}, \mu^M) \leq (\{ \delta_{u^{M,M,N}(t)} \}, \mu^M) + \phi(\mu^M, \mu) \leq 2/M$, and $\{ u^{M,M} \}$ is a generating sequence for $\mu$. \hfill \blacksquare

c) **Generalized gradient controls (“gradient Young measures”).**

We will closer investigate those generalized controls, which are generated by sequences of gradients $(n = 1)$ resp. Jacobi matrices $(n > 1)$. They form a sequentially compact subset $G(K)$ of $Y(K).

**Definition 2.13.** (Generalized gradient controls, “gradient Young measures”):\textsuperscript{30} A measure-valued map $\mu \in Y(K)$ is called a generalized gradient control if it is generated by a sequence $\{ Jx^N \}$, $L^\infty(\Omega, R^{m \times n})$ with $x \in W^{1,\infty}(\Omega, R^n)$ and $Jx^N(t) \in K (\forall) t \in \Omega \forall N \in \mathbb{N}$. The set of (equivalence classes of) generalized gradient controls will be denoted by $G(K) \subseteq Y(K).

**Remarks.**

a) In the literature, the elements of $G(K)$ are referred to as “gradient Young measures” or “gradient parametrized measures”. This notion has been avoided for the same reasons as mentioned after Definition 2.5. above.\textsuperscript{31}
b) Assume that a generating sequence satisfies $x^N \rightharpoondown C^0(\Omega, R^n) x \in W^{1,\infty}(\Omega, R^n)$ and $Jx^N \rightharpoonup L^\infty(\Omega, R^{m \times n})$ $Jx \in L^\infty(\Omega, R^{m \times n})$. If we replace in (P) the Jacobi matrix $Jx$ by a formal control variable $u$ then we obtain a (weakly formulated) state equation $Jx = u$. Then from Definition 2.10., it follows that
\[
\frac{\partial x^N_i}{\partial t_j}(t) = u^N_{ij}(t) \quad (\forall) t \in \Omega \iff \int_{\Omega} \psi_i(t) \left( \frac{\partial x^N_j}{\partial t_i}(t) - u^N_{ij}(t) \right) dt = 0 \quad (\forall) \psi_i \in C^0(\Omega, R) \quad \Longrightarrow \tag{2.31}
\]
\[
\lim_{N \to \infty} \int_{\Omega} \psi_i(t) \left( \frac{\partial x^N_j}{\partial t_i}(t) - \int_{K} v_{ij} d\delta_{(\psi,N)}(t) \right) dt = \int_{\Omega} \psi_i(t) \left( \frac{\partial x_j}{\partial t_i}(t) - \int_{K} v_{ij} d\mu(v) \right) dt = 0 \quad (\forall) \psi_i \in C^0(\Omega, R). \tag{2.32}
\]

\textsuperscript{30} [Kinderlehrer/Pedregal 91], p. 333, [Müller 99], p. 126, Definition 4.1.

\textsuperscript{31} In Wagner 06a, p. 54, Definition 4.13., the German notion “verallgemeinerte Jacobi-Steuerungen” has been proposed.
Consequently, the elements of $\mathcal{G}(K)$ are those generalized controls, which may appear on the right-hand side of the relaxed state equation of (P). Moreover, under the assumptions mentioned above, they satisfy the integrability conditions

$$\int_\Omega \left( \frac{\partial x_i}{\partial t_j}(t) \frac{\partial \psi_i}{\partial h_k}(t) - \frac{\partial x_i}{\partial h_k}(t) \frac{\partial \psi_i}{\partial t_j}(t) \right) dt = \int_\Omega \int_K \left( \frac{\partial \psi_i}{\partial h_k}(t) v_{ij} - \frac{\partial \psi_i}{\partial t_j}(t) v_{ik} \right) d\mu_t(v) dt = 0$$

\[\forall \psi_i \in C^\infty_0(\Omega, \mathbb{R}) \] (2.33)

in the distributional sense. \(^{(32)}\)

**Theorem 2.14. (Existence of generating sequences for generalized gradient controls)**

1) Every sequence $\{x^N\}, W^{1,\infty}(\Omega, \mathbb{R}^n)$ with $\|x^N\|_{L^\infty(\Omega, \mathbb{R}^n)} \leq C$, $Jx^N(t) \in K (\forall) t \in \Omega \forall N \in \mathbb{N}$ admits a subsequence $\{x^{N'}\}$ with $x^{N'} \rightarrow^{C^0(\Omega, \mathbb{R}^n)} x \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ and $Jx^{N'} \rightharpoonup L^\infty(\Omega, \mathbb{R}^{nm}) Jx \in L^\infty(\Omega, \mathbb{R}^{nm})$. Consequently, $\{Jx^{N'}\}$ generates a generalized gradient control $\mu \in \mathcal{G}(K)$.

2) The subset $\mathcal{G}(K) \subset \mathcal{Y}(K)$ of the generalized gradient controls is sequentially compact.

**Proof of Theorem 2.14.** 1) Since the sequences $\{x^N\}$ and $\{Jx^N\}$ are bounded in $L^\infty$-norm, they admit weak*-convergent subsequences $\{x^{N'}\} \rightharpoonup L^\infty(\Omega, \mathbb{R}^n) x$ resp. $\{Jx^{N'}\} \rightharpoonup L^\infty(\Omega, \mathbb{R}^{nm}) y$ with $y = Jx$. By [Dacorogna 04], p. 36, Corollary 1.45., we obtain the norm convergence $x^{N'} \rightarrow L^\infty(\Omega, \mathbb{R}^n)$ $x$, which implies uniform convergence $x^{N'} \rightarrow C^0(\Omega, \mathbb{R}^n) x$ since the functions $x^{N'}$ are continuous. By Theorem 2.11., 1), we may choose $\{x^{N'}\}$ in such a way that $\{Jx^{N'}\}$ generates a generalized gradient control $\mu \in \mathcal{G}(K)$.

2) Assume that a sequence $\{\mu^M\}$, $\mathcal{G}(K)$ converges to $\mu \in \mathcal{Y}(K)$. Since every $\mu^M$ is generated by a sequence $\{Jx^{M,N}\}$, we conclude with Lemma 2.12. that $\mu$ can be generated by a diagonal sequence $\{Jx^{M,N(M)}\}$. Thus $\mu$ is a generalized gradient control as well, and the set $\mathcal{G}(K)$ is closed. Its sequential compactness follows from Theorem 2.7. \(\blacksquare\)

**d) The mean value theorems of Kinderlehrer/Pedregal.**

Shortly spoken, these theorems assign to any generalized control $\mu \in \mathcal{Y}(K)$ resp. $\mu \in \mathcal{G}(K)$ a probability measure $\nu \in rca^{pr}(K)$ (the “average of $\mu$”), which satisfies the variational equality

$$\int_\Omega \int_K g(v) d\mu_t(v) dt = \int_\Omega \int_K g(v) d\nu(v) dt \quad \forall \in C^0(K, \mathbb{R}).$$

(2.34)

More precisely, the following assertions hold:

**Theorem 2.15. (Mean value theorem for generalized controls)** \(^{(33)}\) Assume that $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain with $\vartheta \in \text{int}(\Omega)$. If a sequence $\{u^N\}, L^\infty(\Omega, \mathbb{R}^{nm})$ satisfies

a) $u^N(t) \in K (\forall) t \in \Omega \forall N \in \mathbb{N},$ and

b) $\{u^N\}$ generates a generalized control $\mu \in \mathcal{Y}(K)$,

then there exists a further sequence $\{\tilde{u}^N\}, L^\infty(\Omega, \mathbb{R}^{nm})$ with the following properties:

1) $\tilde{u}^N(t) \in K (\forall) t \in \Omega \forall N \in \mathbb{N},$

\(\text{Cf. [Wagner 99], p. 169 f., Theorem 1.4.}\)

\(\text{Reformulation of [Pedregal 97], p. 117, Theorem 7.1., in terms of generating sequences.}\)

\(^{(32)}\) Cf. [Wagner 99], p. 169 f., Theorem 1.4.

\(^{(33)}\) Reformulation of [Pedregal 97], p. 117, Theorem 7.1., in terms of generating sequences.
2) $\{\tilde{w}^N\}$ generates a constant generalized control $\nu = \{\nu\} \in \mathcal{Y}(K)$, which may be understood as the average of $\mu$ with respect to $t$:

$$
\lim_{N \to \infty} \int_{\Omega} g(w^N(t)) \, dt = \int_{\Omega} \int_{K} g(v) \, d\mu(v) \, dt
$$

(2.35)

Then there exists a sequence of Lipschitz functions $\{\tilde{x}^N\}, W^1_0(\Omega, \mathbb{R}^n)$, with the following properties:

1) $\lim_{N \to \infty} \|\tilde{x}^N\|_{C^0(\Omega, \mathbb{R}^n)} = 0$,

2) $w^N + J\tilde{x}^N(t) \in K \forall t \in \Omega \forall N \in \mathbb{N}$,

3) $\{w^N + J\tilde{x}^N\}$ generates a constant generalized gradient control $\nu = \{\nu\} \in \mathcal{G}(K)$, which may be understood as the average of $\mu$ with respect to $t$:

$$
\lim_{N \to \infty} \int_{\Omega} g(w^N + J\tilde{x}^N(t)) \, dt = \int_{\Omega} \int_{K} g(v) \, d\mu(v) \, dt
$$

(2.36)

4) It holds: $w = \left( \begin{array}{c} \int_{K} v_{11} \, d\nu(v) \\ \vdots \\ \int_{K} v_{nm} \, d\nu(v) \end{array} \right)$.

(2.37)

Theorem 2.15. justifies the definition of an average operator, which assigns to any generalized control a probability measure as its $t$-average. We will prove in Theorem 2.18. below that this operator is continuous.

Definition 2.17. (Average operator for generalized controls) Assume that $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain with $\sigma \in \text{int}(\Omega)$. We define an operator $A: \mathcal{Y}(K) \to \text{rca}^{pr}(K)$, which assigns to every generalized control $\mu \in \mathcal{Y}(K)$ its average $A(\mu) = \nu$ according to Theorem 2.15.

Theorem 2.18. (Continuity of the average operator $A$) Assume again that $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain with $\sigma \in \text{int}(\Omega)$. We endow $\text{rca}^{pr}(K)$ with the distance function $\sigma$ from Definition 2.1 and $\mathcal{Y}(K)$ with the distance function $g$ from Definition 2.8.

1) For all $\mu', \mu'' \in \mathcal{Y}(K)$ it holds that

$$
\sigma(A(\mu'), A(\mu'')) \leq g(\mu', \mu'')
$$

(2.38)

---

34) [KINDERLEHRER/PEDREGAL 91], p. 333: “homogeneous (parametrized) measure”.

35) Generalization of ibid., p. 334, Theorem 2.1.

36) Ibid., p. 336 f.

37) Ibid., p. 337, Proposition 2.2.
In particular, we have the implication
\[ \vartheta(\mu^N, \mu) \to 0 \implies \sigma(A(\mu^N), A(\mu)) \to 0, \] (2.39)
and the average operator \( A \) is continuous with respect to the introduced topologies.

2) The measures of the shape \( \nu = A(\{ \delta_{u(t)} \}) \) with \( u \in L^\infty(\Omega, \mathbb{R}^m) \), \( u(t) \in K (\forall) t \in \Omega \), are dense in the set \( \{ A(\mu) \in \text{rea}^{pt}(K) \mid \mu \in \mathcal{Y}(K) \} \) with respect to its weak* topology (which is generated by the distance function \( \sigma \)).

We continue with a complete proof of Theorem 2.16.38)

**Proof of Theorem 2.16.**39) • **Step 1. Two theorems from measure theory.**

**Lemma 2.19. (Strongly Lipschitz domains are squarable)**40) If \( \Omega \subset \mathbb{R}^m \) is the closure of a strongly Lipschitz domain in the sense of [Morrey 66], p. 72, Definition 3.4.1., then \( \partial \Omega \) is a \( m \)-dimensional Lebesgue null set.

**Theorem 2.20. (Vitali covering theorem)**41) Consider two sets \( \Omega \subset \mathbb{R}^m \) and \( G \subset \mathbb{R}^m \) where \( G \) is a compact set of positive Lebesgue measure with \( v_0 \in \text{int}(G) \). Let \( \mathcal{G} \) be a family, consisting of sets which have been obtained from \( G \) by dilatations with center \( v_0 \) and translations. Moreover, assume that \( \mathcal{G} \) has the property
\[ (*) \quad \text{For almost all } t \in \text{int}(\Omega) \text{ and for any } \varepsilon > 0, \text{ there exists a set } G(t, \varepsilon) \in \mathcal{G} \text{ with } t \in G(t, \varepsilon) \text{ and } \text{Diam}(G(t, \varepsilon)) < \varepsilon. \]

The \( \mathcal{G} \) contains an at most countable subfamily \( \mathcal{G}' \subset \mathcal{G} \) of mutually disjoint sets \( G_1, G_2, \ldots \subseteq \text{int}(\Omega) \) with \( |\text{int}(\Omega) \setminus \bigcup_{i=1}^\infty G_i| = 0. \)

• **Step 2. Definition of the functions \( \tilde{x}^N \).** For every index \( N \in \mathbb{N} \), we define a set family \( \mathcal{G}_N = \{ t + \varepsilon \Omega \mid t \in \Omega, \varepsilon \leq 1/N, t + \varepsilon \Omega \subseteq \Omega \} \), which obviously possesses the property \( (*) \) from the Vitali covering theorem (Theorem 2.20.). Since \( |\partial \Omega| = 0 \) (Lemma 2.19.), we may find sequences \( \{ t^{N,K} \}, \Omega \) and \( \{ \varepsilon^{N,K} \}, \{ 0, 1/N \} \) such that for every fixed \( N \in \mathbb{N} \), \( \Omega \) admits a representation
\[ \Omega = \bigcup_{K=1}^\infty \Omega^{N,K} \cup E^N \] (2.40)
by means of the mutually disjoint sets \( \Omega^{N,K} = t^{N,K} + \varepsilon^{N,K} \Omega \) where \( E^N \) is a null set. Furthermore, we have
\[ |\Omega| = \sum_{K=1}^\infty (\varepsilon^{N,K})^m |\Omega|, \text{ consequently } \sum_{K=1}^\infty (\varepsilon^{N,K})^m = 1. \] (2.41)

Note that \( |\partial \Omega| = 0 \) implies \( \bigcup_{K=1}^\infty \partial \Omega^{N,K} = 0 \) for every \( N \in \mathbb{N} \) as well. Let us define now the functions \( \tilde{x}^N : \Omega \to \mathbb{R}^n \) by
\[ \tilde{x}^N(t) = \begin{cases} \varepsilon^{N,K} \cdot x^N \left( \frac{t - t^{N,K}}{\varepsilon^{N,K}} \right) & \text{if } t \in \text{int}(\Omega^{N,K}); \\ 0 & \text{if } t \in \bigcup_{K=1}^\infty \partial \Omega^{N,K} \cup E^N. \end{cases} \] (2.42)

---

38) The complete proof of Theorem 2.15. may be found in [Wagner 06a], p. 57 f., while [Pedregal 97], p. 117 f., gives only a sketch.

39) Cf. [Kinderlehrer/Pedregal 91], p. 335 f.

40) [Wagner 07c], p. 9, Lemma 3.1.

41) [Dacorogna/Marcellini 99], p. 231 f., Corollary 10.6.
**Step 3. Proof of Assertions 1) and 2.** Since $\bigcup_{K=1}^{\infty} \text{int}(\Omega^{N,K}) \subseteq \text{int}(\Omega)$, $\tilde{x}^N$ takes the boundary value $\tilde{x}^N(t) = o$ for all $t \in \partial \Omega$. In order to prove the continuity of $\tilde{x}^N$, it suffices to inspect its behaviour along a sequence of points $\{ s_n \} \subseteq \text{int}(\Omega^{N,K}) \to t_0 \in \partial \Omega^{N,K}$. Then, from the continuity of $x^N$, it follows that

$$
\lim_{s \to s_n} x^N(s) = x^N(t_0) = o.
$$

For all $t \in \bigcup_{K=1}^{\infty} \partial \Omega^{N,K} \cup E^N$, $\tilde{x}^N(t) = o$ from the outset, and for arbitrary $t \in \bigcup_{K=1}^{\infty} \text{int}(\Omega^{N,K})$, the estimate

$$
|\tilde{x}^N(t)| = \left| \sum_{K=1}^{\infty} \mathbb{1}_{\text{int}(\Omega^{N,K})}(t) \cdot x^N \left( \frac{t - t^{N,K}}{\varepsilon^{N,K}} \right) \right| \leq \sum_{K=1}^{\infty} \mathbb{1}_{\text{int}(\Omega^{N,K})}(t) \cdot \frac{1}{N} \cdot \left| x^N \left( \frac{t - t^{N,K}}{\varepsilon^{N,K}} \right) \right| \quad (2.44)
$$

holds. By assumption b), the functions $x^N$ are uniformly bounded, and we arrive at 1). Together with $x^N$, $\tilde{x}^N$ is differentiable a.e. on $\Omega$ with

$$
J\tilde{x}^N(t) = Jx^N \left( \frac{t - t^{N,K}}{\varepsilon^{N,K}} \right) \quad (\forall) \ t \in \Omega^{N,K} \ \forall \ N \in \mathbb{N},
$$

and $w^N + J\tilde{x}^N(t) \in K$ implies $w^N + J\tilde{x}^N(t) \in K$ for almost all $t \in \Omega$. Consequently, the functions $\tilde{x}^N$ satisfy 2) as well and belong to $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$.

**Step 4. Proof of Assertion 3.** Let us investigate now whether $\{ w^N + J\tilde{x}^N \}$ generates a generalized control. Choose $f \in C^0(\Omega, \mathbb{R}) \subset L^1(\Omega, \mathbb{R})$ and $g \in C^0(\mathbb{R}, \mathbb{R})$ where $g$ is bounded from below on $K$ through $C \leq g(v)$ for all $v \in K$. Then it holds that

$$
\int_{\Omega} f(t) g(w^N + J\tilde{x}^N(t)) \, dt = \sum_{K=1}^{\infty} \int_{\Omega^{N,K}} f(t) g(w^N + Jx^N \left( \frac{t - t^{N,K}}{\varepsilon^{N,K}} \right)) \, dt
$$

$$
= \sum_{K=1}^{\infty} \int_{\Omega} \left( \varepsilon^{N,K} \right)^m f(t^{N,K} + \varepsilon^{N,K} t_{N,K}^{\prime}) \cdot \int_{\Omega} f(t^{N,K} + \varepsilon^{N,K} t_{N,K}^{\prime}) \cdot \int_{\Omega} \left( g(w^N + Jx^N(\tau)) - C \right) \, d\tau
$$

$$
= \sum_{K=1}^{\infty} \left( \varepsilon^{N,K} \right)^m \cdot \left| \Omega \right| \cdot f(t^{N,K} + \varepsilon^{N,K} t_{N,K}^{\prime}) \cdot \int_{\Omega} g(w^N + Jx^N(\tau)) \, d\tau
$$

$$
+ \left( \varepsilon^{N,K} \right)^m \cdot \left| \Omega \right| \cdot f(t^{N,K} + \varepsilon^{N,K} t_{N,K}^{\prime}) \cdot \int_{\Omega} g(w^N + Jx^N(\tau)) \, d\tau
$$

$$
- C \left( \varepsilon^{N,K} \right)^m \cdot \left| \Omega \right| \cdot f(t^{N,K} + \varepsilon^{N,K} t_{N,K}^{\prime}),
$$

where the mean value theorem ([Elstrodt 96], p. 154, Theorem 6.6.) has been applied to the integrands

$$
f(i^{N,K} + \varepsilon^{N,K} t_{N,K}^{\prime}) \cdot 1 \quad \text{and} \quad f(t^{N,K} + \varepsilon^{N,K} t_{N,K}^{\prime}) \cdot \left( g(w^N + Jx^N(\tau)) - C \right).
$$

From $t_{N,K}^{\prime}, t_{N,K}^{\prime\prime} \in \Omega$ it follows

$$
i^{N,K} + \varepsilon^{N,K} t_{N,K}^{\prime} \in \Omega^{N,K} \quad \text{resp.} \quad i^{N,K} + \varepsilon^{N,K} t_{N,K}^{\prime\prime} \in \Omega^{N,K}.
$$

The first and the third series in (2.48) form (countable) Riemann sums for the continuous integrand $f$ over the squarable set $\Omega$ with respect to the decomposition $\Omega = \bigcup_{k=1}^{\infty} \Omega^{N,K} \cup E^N$ with fineness $\leq 1/N$; consequently, for $N \to \infty$ they both converge to $C \cdot \int_{\Omega} f(t) \, dt$. By Theorem 2.14., 1), a subsequence of
\{ \tilde{w}^N + J \tilde{x}^N \} (we keep the index \( N \)) generates a generalized gradient control \( \nu = \{ \nu_t \} \in \mathcal{G}(K) \). It follows that

\[
\lim_{N \to \infty} \int_{\Omega} f(t) g(\tilde{w}^N + J \tilde{x}^N(t)) \, dt = \int_{\Omega} \int_{K} f(t) g(v) \, d\nu_t(v) \, dt = \frac{1}{|\Omega|} \int_{\Omega} \int_{K} f(t) \, dt \cdot \int_{\Omega} \int_{K} g(v) \, d\mu_r(v) \, dr. \quad (2.51)
\]

This identity may be extended to arbitrary \( f \in L^1(\Omega, \mathbb{R}) \). In order to show that \( \nu \) is a.e. constant, we fix \( t_0 \in \text{int}(\Omega) \) and define for every \( \delta > 0 \) a function \( f_\delta \in L^1(\Omega, \mathbb{R}) \) by

\[
f_\delta(t) = \frac{1}{|K(t_0, \delta)|} \left| K(t_0, \delta) \right|
\]

where \( K(t_0, \delta) \subset \Omega \) denotes a closed ball with radius \( \delta \) and center \( t_0 \). Insertion into (2.51) yields

\[
\int_{\Omega} \int_{K} f_\delta(t) g(v) \, d\nu_t(v) \, dt = \frac{1}{|\Omega|} \int_{\Omega} \int_{K} f(t) \, dt \cdot \int_{\Omega} \int_{K} g(v) \, d\mu_r(v) \, dt = \frac{1}{|\Omega|} \int_{\Omega} \int_{K} g(v) \, d\mu_r(v) \, dt.
\]

Since the function \( h_\delta(t) = \int_{K} g(v) \, d\nu_t(v) \) is measurable and essentially bounded, we may pass to the limit with \( \delta \to 0 \) for almost all \( t_0 \in \Omega \), and we arrive at

\[
\lim_{\delta \to 0} \frac{1}{|K(t_0, \delta)|} \left| K(t_0, \delta) \right| \int_{K(t_0, \delta)} \left[ \int_{K} g(v) \, d\nu_t(v) \right] \, dt = \int_{K} g(v) \, d\nu_{t_0}(v) = \frac{1}{|\Omega|} \int_{\Omega} \int_{K} g(v) \, d\mu_r(v) \, dt \quad \forall g \in C^0(K, \mathbb{R}).
\]

(2.54)

Consequently, the linear, continuous functionals \( \nu_{t_0} \in rca^{pr}(K) \subset \left( C^0(K, \mathbb{R}) \right)^* \) coincide with a single element \( \nu \) for almost all \( t_0 \in \Omega \), and by insertion of \( f(t) \equiv 1 \), we get 3).

**Step 5. Proof of Assertion 4.** In order to prove 4), we apply the Gauss' theorem ([Evans/Gariepy 92], p. 133, Theorem 1, (ii)) to \( f(t) = y^N(t) = \tilde{x}^N(t) + (w^N(t), \varphi_1(t), \ldots, \varphi_{j-1}(t), \varphi_j(t), \ldots, \varphi_m(t)) \equiv 0 \) and \( \varphi_j(t) \equiv 1 \):

\[
\int_{\Omega} y^N(t) \, d\varphi(t) + \int_{\Omega} \sum_{k=1}^{m} \frac{\partial y^N}{\partial t_j}(t) \varphi_k(t) \, dt = \int_{\Omega} \sum_{k=1}^{m} n_k(s) \varphi_k(s) y^N(s) \, ds \implies (2.55)
\]

\[
\int_{\Omega} \frac{\partial y^N}{\partial t_j}(t) \, dt = \int_{\Omega} n_j(t) y^N(t) \, dt = \int_{\Omega} n_j(s) \sum_{k=1}^{m} w_{ik} t_k \, ds
\]

\[
= \int_{\Omega} \left( \sum_{k=1}^{m} w_{ik} t_k \right) \, d\varphi(t) + \int_{\Omega} \sum_{k=1}^{m} \frac{\partial}{\partial t_k} \left( \sum_{k=1}^{m} w_{ik} t_k \right) \varphi_k(t) \, dt = \int_{\Omega} w^N_{ij} \, dt.
\]

Insert now \( g(v) = v_{ij} \) for all \( 1 \leq i \leq n, 1 \leq j \leq m \). Then from Part 3), it follows that

\[
\lim_{N \to \infty} \int_{\Omega} g(w^N + J \tilde{x}^N(t)) \, dt = \lim_{N \to \infty} \int_{\Omega} \left( w_{ij}^N + \frac{\partial \tilde{x}^N}{\partial t_j}(t) \right) \, dt
\]

\[
= \lim_{N \to \infty} \int_{\Omega} \frac{\partial y^N}{\partial t_j}(t) \, dt = \lim_{N \to \infty} \int_{\Omega} w^N_{ij} \, dt = \int_{\Omega} w_{ij} \, dt = \int_{\Omega} \int_{K} v_{ij} \, d\nu(v) \, dt.
\]

Dividing by \( |\Omega| \), we arrive at 4), and the proof is complete. \( \blacksquare \)
Proof of Theorem 2.18. 42) 1) Put \( \nu' = A(\mu') \) and \( \nu'' = A(\mu'') \). From the definitions of \( \sigma \) and \( \varrho \), it follows together with Theorem 2.15.:

\[
\sigma(\nu', \nu'') = \sum_{s=1}^{\infty} \frac{1}{2^{1+s} (1 + L_s)} \left| \int_K g_s(v) \left( dv'(v) - dv''(v) \right) \right| \quad (2.58)
\]

\[
= \sum_{s=1}^{\infty} \frac{1}{2^{1+s} (1 + L_s)} \left| \int_{\Omega} \int_K g_s(v) \left( dv'(v) - dv''(v) \right) dt \right| \quad (2.59)
\]

\[
= \sum_{s=1}^{\infty} \frac{1}{2^{1+s} (1 + L_s)} \left| \int_{\Omega} \int_K g_s(v) \left( dp'_t(v) - dp''_t(v) \right) dt \right| \quad (2.60)
\]

\[
\leq \sum_{s=1}^{\infty} \frac{1}{2^{1+s} (1 + L_s)} \left| \int_{\Omega} \int_K g_s(v) \left( dp'_t(v) - dp''_t(v) \right) dt \right| + \sum_{s=2}^{\infty} \sum_{r=1}^{\infty} \frac{1}{2^{r+s} (1 + L_s)} \left| \int_{\Omega} \int_K f_s(t) g_s(v) \left( dp'_t(v) - dp''_t(v) \right) dt \right| = \varrho(\nu', \nu'').
\]

2) Let \( \nu = A(\mu) \) be a probability measure, which results as the average of a generalized control \( \mu \in \mathcal{Y}(K) \). By Theorem 2.11., 2), there exists a sequence \( \{ \mu^N \} \subset \mathcal{Y}(K) \) with \( \mu^N \to \mu \) and \( \mu^N \in L^\infty(\Omega, \mathbb{R}^m) \), \( w^N(t) \in K (\forall t \in \Omega \land N \in \mathbb{N}) \) and from Part 1), we get \( A(\mu^N) \rightharpoonup A(\mu) = \nu. \)

3. The set-valued maps \( S^* \) and \( S^# \).

Throughout the whole section, we assume that \( \Omega \subset \mathbb{R}^m \) is the closure of a strongly Lipschitz domain with \( \phi \in \text{int}(\Omega) \), what guarantees the applicability of Theorem 2.16. Further, we fix a convex body \( K \subset \mathbb{R}^m \) with \( \phi \in \text{int}(K) \) and the quantities \( c_K = \text{Dist}(\phi, \partial K) \) and \( C_K = \max \{ 1, \max_{v \in K} |v| \} \), thus \( 0 < c_K \leq C_K \) and \( \text{Diam}(K) \leq 2C_K \).

a) The set-valued map \( S^* \).

We start with the definition of a set-valued map \( w \mapsto S^*(w) \subset \text{rca}^{pr}(K) \) on the points \( w \in \text{int}(K) \). \( S^* \) possesses nonempty, convex, weak*-sequentially compact images, whose elements result as averages of generalized gradient controls.

Definition 3.1. (Definition of \( S^*(w) \) for \( w \in \text{int}(K) \)) For \( w \in \text{int}(K) \), we define the following set of probability measures:

\[
S^*(w) = \left\{ \nu \in \text{rca}^{pr}(K) \mid \text{there exist sequences } \{ w^N \} \subset \text{rca}^1(\Omega, \mathbb{R}^m) \text{ and } \{ x^N \} \subset W_0^{1,\infty}(\Omega, \mathbb{R}^n) \text{ with}
\right. \]

\[
a) \lim_{N \to \infty} w^N = w,
\]

\[
b) \lim_{N \to \infty} \| x^N \|_{C^0(\Omega, \mathbb{R}^n)} = 0,
\]

\[
c) w^N + Jx^N(t) \in K (\forall t \in \Omega \land N \in \mathbb{N}),
\]

\[
d) \{ w^N + Jx^N \} \text{ generates the constant generalized gradient control } \nu = \{ \nu \}. \right\}
\]

Lemma 3.2. (Special generating sequences in Definition 3.1.) Let \( w \in \text{int}(K) \). For every \( \nu \in S^*(w) \), there exist sequences \( \{ w^N \} \) and \( \{ x^N \} \) with the properties a) - d) from Definition 3.1., which satisfy additionally \( w^N \in \text{int}(K) \) as well as \( w^N + Jx^N(t) \in \text{int}(K) (\forall t \in \Omega \land N \in \mathbb{N}) \).

Lemma 3.3. (Dense subset of \( S^*(w) \)) If \( w \in \text{int}(K) \), \( \delta_w \in S^*(w) \), and the measures of the shape \( \nu = A(\delta_{(w+Jx(t))}) \), obtained from functions \( x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n) \) with \( w + Jx(t) \in K (\forall t \in \Omega \land N \in \mathbb{N}) \) are dense in \( S^*(w) \) with respect to the weak* topology (resp. the metric \( \sigma \) from Definition 2.1.).

\[\text{Cf. [Kinderlehrer/Pedregal 91, p. 337.]}\]
Theorem 3.4. (Properties of the sets $S^*(w)$) For every $w \in \text{int}(K)$, the set $S^*(w) \subseteq \text{rca}^{pr}(K)$ is nonempty, convex and weak*-sequentially compact.

The further investigation of the set-valued map $S^*$ runs parallel to the investigation of the envelope $f^*$ in [WAGNER 07A], Sections 3.b) and 3.c). In the present subsection, we start with the proof of the continuity of $S^*$ on int $(K)$, which is based on a $\varepsilon$-$\delta$ relation depending on the distance of the given points $v, w \in \text{int}(K)$ as well as on their distances to the boundary $\partial K$.

Theorem 3.5. ($\varepsilon$-$\delta$ relation for $S^*$) For every $0 < \varepsilon < 1$ there exists $\delta_1(\varepsilon) = \frac{1}{2} \varepsilon/C_K > 0$ such that for all $v, w \in \text{int}(K)$, the following $\varepsilon$-$\delta$ relation holds:

$$|v - w| \leq \delta_1(\varepsilon) \cdot \text{Min} \left(1, \text{Dist}(v, \partial K), \text{Dist}(w, \partial K)\right) \implies H(S^*(v), S^*(w)) \leq \varepsilon \quad (3.2)$$

where $C_K$ is the quantity defined in the beginning of the section, and $H(\cdot, \cdot)$ denotes the Hausdorff distance (cf. Definition 5.2. in the Appendix).

Theorem 3.6. (Continuity of the set-valued map $S^*$) The set-valued map $S^*$ is continuous on int $(K)$.

Proof of Lemma 3.2. For arbitrary $\nu \in S^*(w)$ there exist sequences $\{w^n\}$, $K$ and $\{x^n\}, W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with the properties a) – d) from Definition 3.1. Choose now a sequence of numbers $\{c^n\}, \mathbb{R}$ with $0 < c^n < 1$ for all $n \in \mathbb{N}$ and $\lim_{N \rightarrow \infty} c^n = 1$. Obviously, the sequences $\{c^n w^n\}$ and $\{c^n x^n\}$ possess the properties a) – d) from Definition 3.1. as well, while $c^n w^n \in \text{int}(K)$ as well as $c^n (w^n + J x^n(t)) \in \text{int}(K)$ for all $t \in \Omega$.

It holds further that

$$g(\{\delta_n(w^n + J x^n(t))\}, \{\nu\}) \leq g(\{\delta_n(w^n + J x^n(t))\}, \{\nu\}) + g(\{\delta_n(w^n + J x^n(t))\}, \{\delta_n(w^n + J x^n(t))\}) \quad (3.3)$$

where, by definition of $g$, the first member converges to zero, while the second member can be estimated by

$$g(\{\delta_n(w^n + J x^n(t))\}, \{\delta_n(w^n + J x^n(t))\}) \leq \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s} (1 + L_s)} \int_{\Omega} f_r(t) \left(g_s(c^n w^n + c^n J x^n(t)) - g_s(c^n w^n + J x^n(t))\right) dt \quad (3.4)$$

Since $\|f_r\|_{L^1(\Omega, \mathbb{R})} \leq 1$ for all $r \in \mathbb{N}$ and $w^n + J x^n(t) \in K$ for all $t \in \Omega$, the second member converges to zero as well. Consequently, $\nu \in S^*(w)$ can be generated in the claimed way.

Proof of Lemma 3.3. Let $w \in \text{int}(K)$ be given. We observe first that the constant sequences $\{w^n\}$ and $\{x^n\}$ with $w^n = w$ and $x^n \equiv w$ satisfy a) – c) from Definition 3.1., and $\{w^n + J x^n\}$ generates the constant generalized gradient control $\mu = \{\delta_w\}$. Thus $\delta_w \in S^*(w)$. For arbitrary $\nu \in S^*(w)$, we choose sequences $\{w^n\}$ and $\{x^n\}$ with the properties a) – d) from Definition 3.1. We have $\text{Dist}(w, \partial K) = C > 0$ since $w \in \text{int}(K)$. Define now the functions

$$y^n = \frac{C}{C + \|w^n - w\|} x^n \in W_0^{1,\infty}(\Omega, \mathbb{R}^n). \quad (3.6)$$

43) Cf. [WAGNER 07A], p. 16, Theorem 3.5.

44) Cf. ibid., p. 17, Theorem 3.6., 1).
For all $N \in \mathbb{N}$ we obtain:

\[ w^N + Jx^N(t) \in K \quad (\forall) \ t \in \Omega \quad \implies \quad w + Jx^N(t) \in K + K(\omega, |w^N - w|) \quad (\forall) \ t \in \Omega \quad \implies (3.7) \]

\[ w + \frac{C}{C + |w^N - w|} Jx^N(t) = w + Jy^N(t) \in K \quad (\forall) \ t \in \Omega. \tag{3.8} \]

By definition of $\nu$, the first member within the inequality

\[ \varrho(\{ \delta_{w + Jy^N(t)} \} ; \{ \nu \}) \leq \varrho(\{ \delta_{w^N + Jx^N(t)} \} ; \{ \nu \}) + \varrho(\{ \delta_{w + Jy^N(t)} \} , \{ \delta_{w + Jy^N(t)} \}) \tag{3.9} \]

converges to zero, and the second member obeys the estimate

\[ \varrho(\{ \delta_{w^N + Jx^N(t)} \} , \{ \delta_{w + Jy^N(t)} \}) \]

\[ = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s}(1+L_s)} \left| \int_{\Omega} f_r(t) \left( g_s(w^N + Jx^N(t)) - g_s(w + \frac{C}{C + |w^N - w|} Jx^N(t)) \right) dt \right| \tag{3.10} \]

\[ \leq \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s}(1+L_s)} \int_{\Omega} f_r(t) \cdot L_s \cdot \left| w - w^N + (1 - \frac{C}{C + |w^N - w|}) Jx^N(t) \right| dt \tag{3.11} \]

\[ \leq \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s}} \cdot \frac{L_s}{1+L_s} \cdot \|f_r\|_{L^1(\Omega;R)} \cdot \left( \|w^N - w\| + \frac{|w^N - w|}{C + |w^N - w|} \cdot \text{ess sup}_{t \in \Omega} \|Jx^N(t)\| \right). \tag{3.12} \]

By $\|f_r\|_{L^1(\Omega;R)} \leq 1$ $\forall r \in \mathbb{N}$ (Definition 2.8.) and the uniform essential boundedness of $Jx^N(t)$, this member converges zero as well. By Theorem 2.18, 1), from $\varrho(\{ \delta_{w + Jy^N(t)} \} , \{ \nu \}) \to 0$ it follows that $A(\{ \delta_{w + Jy^N(t)} \}) \xrightarrow{a.s.} A(\{ \nu \}) = \nu$. On the other hand, by the mean value theorem (Theorem 2.16.), the probability measures of the shape $A(\{ \delta_{w + Jy^N(t)} \})$ belong to $S^*(w)$ as well. Thus $\nu$ can be approximated in the claimed way. \hspace{1cm} ■

**Proof of Theorem 3.4.**  
**Step 1.** The set $S^*(w)$ is nonempty. By Lemma 3.3., we have $\delta_w \in S^*(w)$ for all $w \in \text{int}(K)$.

**Step 2.** The set $S^*(w)$ is convex. Let $\nu', \nu'' \in S^*(w)$ and $0 < \lambda < 1$ be given; we will prove that the convex combination $\nu = \lambda \nu' + (1 - \lambda) \nu''$ belongs to $S^*(w)$ as well. We choose first a subset $E \subset \text{int}(\Omega)$, which forms the closure of a strongly Lipschitz domain as well, and satisfies $|E| = \lambda |\Omega|$. (E may be obtained e. g. as the image of $\Omega$ under a homothety with center $\omega \in \Omega$.) Define further subsets

\[ E^K = \{ t \in E \mid \text{Dist}(t, \partial E) \geq 1/K \} \tag{3.13} \]

and functions $\eta^K \in W^{1,\infty}(\Omega, \mathbb{R})$ with

\[ \eta^K(t) = \begin{cases} 0 & \forall t \in \Omega \setminus E, \\
1 & \forall t \in E, \\
1 & \forall t \in E^K, \\
0 & \forall t \in E \setminus E^K,
\end{cases} \quad \text{and} \quad |\nabla \eta^K(t)| \leq C_1 \cdot K \quad (\forall) t \in \Omega. \tag{3.14} \]

We will investigate now the measure-valued map

\[ \mu = \{ \mu_1 \} = \{ \mathbb{I}_E(t) \cdot \nu' + \mathbb{I}_{(\Omega \setminus E)}(t) \cdot \nu'' \}. \tag{3.15} \]

Obviously, $\mu$ is a generalized control; we will show that $\mu$ belongs to $\mathcal{G}(K)$. For this purpose, we choose sequences $\{ w_N' \}, K$ and $\{ x'_N \}, W^{1,\infty}(\Omega, \mathbb{R}^n)$ resp. $\{ w''_N \}, K$ and $\{ x''_N \}, W^{1,\infty}(\Omega, \mathbb{R}^n)$ with the properties

45) Cf. [Kinderlehrer/Pedregal 91], p. 339 f.
a) – c) from Definition 3.1. such that \( \{ w'_{N} + Jx'_{N} \} \) and \( \{ w''_{N} + Jx''_{N} \} \) generate the constant generalized gradient controls \( \mu^\prime = \{ \mu^\prime \} \) and \( \mu'' = \{ \mu'' \} \). Define now the functions

\[
\begin{align*}
x^{N,K}(t) &= (\eta^{K}(t)) w'_{N} t + (1 - \eta^{K}(t)) w''_{N} t + \eta^{K}(t) x'_{N}(t) + (1 - \eta^{K}(t)) x''_{N}(t) \quad \text{with} \\
Jx^{N,K}(t) &= \eta^{K}(t)(w'_{N} + Jx'_{N}(t)) + (1 - \eta^{K}(t))(w''_{N} + Jx''_{N}(t))
\end{align*}
\]

(3.16) (3.17)

(the points \( w, w'_{N}, w''_{N} \) have to be understood as \((n,m)\)-matrices). The first both members form a convex combination, which belongs to \( K \) for all \( t \in \Omega \). By assumptions a) and b) from Definition 3.1., we find to every \( K \) an index \( N(K) \) with

\[
\| x'_{N} + w'_{N} t - w t \|_{C^{0}(\Omega; \mathbb{R}^{m})} \leq \frac{1}{2C_{1}K^{2}} \quad \text{and} \quad \| x''_{N} + w''_{N} t - w t \|_{C^{0}(\Omega; \mathbb{R}^{m})} \leq \frac{1}{2C_{1}K^{2}} \quad \forall N \geq N(K).
\]

(3.18)

With a constant \( C_{2} > 0 \), depending on the matrix norm in \( \mathbb{R}^{n \times m} \), it holds that

\[
\left| (x'_{N(K)}(t) + w'_{N} t - w t) + (w t - w''_{N} t - x''_{N(K)}(t)) \right| \cdot \nabla \eta^{K}(t) \leq C_{2} \cdot \left( \| x'_{N} + w'_{N} t - w t \|_{C^{0}(\Omega; \mathbb{R}^{m})} + \| x''_{N} + w''_{N} t - w t \|_{C^{0}(\Omega; \mathbb{R}^{m})} \right) \cdot \| \nabla \eta^{K}(t) \| \leq C_{2}/K;
\]

consequently, \( Jx^{N(K),K}(t) \in K(K, C_{2}/K) \) for all \( t \in \Omega \). With the number \( c_{K} = \text{Dist} (\mathfrak{a}, \partial K) > 0 \), we obtain

\[
\frac{c_{K} K}{c_{K} K + C_{2}} \left( K + K(K, C_{2}/K) \right) \subseteq K.
\]

(3.20)

We claim that the sequences

\[
w^{K} = \frac{c_{K} K}{c_{K} K + C_{2}} \cdot w''_{N(K)} \quad \text{and} \quad y^{K}(t) = \frac{c_{K} K}{c_{K} K + C_{2}} \cdot \left( x^{N(K),K} - w''_{N(K)} t \right)
\]

satisfy the assumptions a) – c) from Theorem 2.16., and that \( \{ w^{K} + Jy^{K} \} \) generates \( \mu \) as well. It is clear that \( w^{K} \in K \) for all \( K \in \mathbb{N} \) and

\[
\lim_{K \to \infty} w^{K} = \lim_{K \to \infty} \frac{c_{K} K}{c_{K} K + C_{2}} \cdot w''_{N(K)} = w.
\]

(3.22)

For all \( t \in \partial \Omega \), we have \( \eta^{K}(t) = 0 \) and \( y^{K}(t) = \frac{c_{K} K}{c_{K} K + C_{2}} \cdot x''_{N(K)}(t) = \mathfrak{a} \); consequently, \( y^{K} \) belongs to \( W^{1,\infty}_{0} (\Omega; \mathbb{R}^{m}) \) together with the functions \( x'_{N} \) and \( x''_{N} \). From the construction above, it follows that \( w^{K} + Jy^{K}(t) \in K (\forall) t \in \Omega \forall K \in \mathbb{N} \) as well. After all, \( \{ w^{K} + Jy^{K} \} \) generates the generalized control \( \mu \).

To prove this, we fix \( f \in C^{0}(\Omega; \mathbb{R}) \subset L^{1}(\Omega, \mathbb{R}) \) and \( g \in C^{0}(\Omega, \mathbb{R}) \) and calculate

\[
\begin{align*}
\int_{\Omega} f(t) g(w^{K} + Jy^{K}(t)) \, dt &= J_{1,K} + J_{2,K} + J_{3,K} \\
J_{1,K} &= \int_{\Omega} f(t) g\left( \frac{c_{K} K}{c_{K} K + C_{2}} \cdot (w'_{N(K)} + Jx'_{N(K),K}(t)) \right) \, dt; \\
J_{2,K} &= \int_{\Omega \setminus E} f(t) g\left( \frac{c_{K} K}{c_{K} K + C_{2}} \cdot (w''_{N(K)} + Jx''_{N(K),K}(t)) \right) \, dt; \\
J_{3,K} &= \int_{E \setminus R^{K}} f(t) g(w^{K} + Jy^{K}(t)) \, dt.
\end{align*}
\]

(3.23) (3.24) (3.25) (3.26)
Passing to the limit $K \to \infty$, we obtain

$$
\lim_{K \to \infty} J_{1,K} = \int_E \int_K f(t) g(v) \, dv' \, dt; \\
\lim_{K \to \infty} J_{2,K} = \int_{\Omega \setminus E} \int_K f(t) g(v) \, dv'' \, dt; \\
\lim_{K \to \infty} J_{3,K} = 0,
$$

(3.27, 3.28, 3.29)

since the integrands $f(t) \cdot g(w^K + J_y^K(t))$ are uniformly bounded on $\Omega$. Consequently, it holds

$$
\lim_{K \to \infty} \int_{\Omega} f(t) g(w^K + J_y^K(t)) \, dt = \int_E \int_K f(t) g(v) \, dv' \, dt + \int_{\Omega \setminus E} \int_K f(t) g(v) \, dv'' \, dt,
$$

(3.30)

and $\{ w^K + J_y^K \}$ generates $\mu$. Let us apply now the mean value theorem (Theorem 2.16.) to $\{ w^K \}$ and $\{ y^K \}$. The constant generalized gradient control $\tilde{\nu} = \{ \tilde{\nu} \}$, which is generated by $\{ w^K + J_y^K \}$, satisfies

$$
\int_{\Omega} g(v) \, d\tilde{\nu}(v) \, dt = \int_E \int_K g(v) \, d\mu(v) \, dt + \int_{\Omega \setminus E} \int_K g(v) \, dv'' \, dt \quad \implies \quad \int_{\Omega} g(v) \, d\tilde{\nu}(v) = \frac{\|E\|}{|\Omega|} \int_{\Omega} g(v) \, dv' + \frac{\|\Omega \setminus E\|}{|\Omega|} \int_{\Omega} g(v) \, dv'' = \int_{\Omega} g(v) \, d\nu(v' + (1 - \lambda) \nu''(v))
$$

(3.31, 3.32)

for all $g \in C^0(K,R)$. We see that $\nu$ and $\tilde{\nu}$ coincide, and $\nu$ belongs to $S^*(w)$.

\begin{itemize}
\item **Step 3.** The set $S^*(w)$ is weak*-sequentially compact. Let a sequence $\{ \nu^N \}, S^*(w) \xrightarrow{\sigma} \nu \in rca(K)$ be given. Since $rca^P(K)$ itself is weak*-sequentially compact, the limit element $\nu$ is a probability measure as well. Consider now the sequence of constant generalized gradient controls $\mu^N = \{ \nu^N \}$. By Theorem 2.14., 2), it admits a subsequence $\{ \mu^{N'} \}$, which converges to a generalized gradient control $\mu \in \mathcal{G}(K)$.

With the aid of Lemma 2.12., from the generating sequences $\{ w^{N'} + J_x^{N',K(N')} \}$ for the $\mu^{N'}$, we may select a diagonal sequence $\{ w^{N'} + J_x^{N',K(N')} \}$ as generating sequence for $\mu$ where $\{ w^{N'} \}$ and $\{ x^{N,K(N')} \}$ possess the properties a) – c) from Definition 3.1. Applying the mean value theorem (Theorem 2.16.) to these sequences, we find a sequence $\{ w^{N'} + J_x^{N',K(N')} \}$, which generates a constant generalized gradient control $\nu' = \{ \nu' \}$. From $\mu^{N'} \to \mu$ and the continuity of the average operator (Theorem 2.18., 1)), it follows that $\nu^{N'} = A(\mu^{N'}) \xrightarrow{\sigma} A(\mu) = \nu'$. Since $\{ \nu^{N'} \}$ is a subsequence of the weak*-convergent sequence $\{ \nu^N \} \xrightarrow{\sigma} \nu$, we arrive at $\nu' = \nu$. Consequently, $\{ \nu \}$ can be generated by sequences with the properties a) – d) from Definition 3.1., $\nu$ belongs to $S^*(w)$, and the set $S^*(w)$ is weak*-closed. By Theorem 2.14., 2), the images $S^*(w)$ are weak*-sequentially compact.
\end{itemize}

**Proof of Theorem 3.5.** Let $0 < \varepsilon < 1$ and $v, w \in \text{int}(K)$ with

$$
|v - w| \leq \frac{\varepsilon}{4c_K} \cdot \text{Min} \{ 1, \text{Dist}(v, \partial K) \}
$$

be given. Then we choose arbitrary $\nu' \in S^*(v)$ and show that there exists $\nu'' \in S^*(w)$ with $\sigma(\nu', \nu'') \leq \varepsilon$ (cf. Definition 5.2.). For $\nu'$ there exist sequences $\{v^N\}, K$ and $\{x^N\}, W^1_0(\Omega, R^n)$ with the properties a) – d) from Definition 3.1. By Lemma 3.2. and 3.3., we may assume that $v^N = v$ and $v + J_x^N(t) \in \text{int}(K)$ $(\forall) t \in \Omega$ for all $N \in \mathbb{N}$. We invoke the following geometrical lemma:

**Lemma 3.7.**

46) Let a nonempty, convex, compact set $K \subset \mathbb{R}^m$ with $v_0 \in \text{int}(K)$ and a function $x \in W^1_0(\Omega, R^n)$ be given. Then it holds for all $0 < \lambda < 1:

$$
v_0 + J_x(t) \in \text{int}(K) \quad (\forall) t \in \Omega \implies \lambda \cdot \text{Dist}(v_0, \partial K) \leq \text{Dist} \left( v_0 + (1 - \lambda) J_x(t), \partial K \right) \quad (\forall) t \in \Omega.
$$

(3.34)

We define \( \delta_1(\varepsilon) = \frac{1}{4} \varepsilon/C_K = \lambda \) and find
\[
v + Jx^N(t) \in \text{int}(K) \quad (\forall) t \in \Omega \quad \implies \quad v + (1 - \lambda) Jx^N(t) \in K \quad (\forall) t \in \Omega \quad \forall N \in \mathbb{N}; \tag{3.35}
\]
From Lemma 3.7., we get for all \( w \in \text{int}(K) \) the implication
\[
|v - w| \leq \lambda \cdot \min \left(1, \text{Dist}(v, \partial K)\right) \leq \text{Dist}(v + (1 - \lambda) Jx^N(t), \partial K) \quad \implies \quad
w + (1 - \lambda) Jx^N(t) = (w - v) + \left(v + (1 - \lambda) Jx^N(t)\right) \in K \quad (\forall) t \in \Omega \quad \forall N \in \mathbb{N}, \tag{3.36}
\]
from which follows that
\[
|v - w| \leq \lambda \cdot \min \left(1, \text{Dist}(v, \partial K)\right) \quad \implies \quad |v + Jx^N(t) - (w + (1 - \lambda) Jx^N(t))| \leq \varepsilon
\quad (\forall) t \in \Omega \quad \forall N \in \mathbb{N}. \tag{3.37}
\]
We obtain for all \( N \in \mathbb{N} \):
\[
g(\{\delta_{v+Jx^N(t)}\}, \{\delta_{w+(1-\lambda)Jx^N(t)}\}) \\
= \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s+1}} \cdot \frac{1}{1 + L_s} \cdot \left| \int_{\Omega} f_r(t) \left( g_s(v + Jx^N(t) - g_s(w + (1 - \lambda) Jx^N(t)) \right) dt \right| \tag{3.38}
\leq \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s+1}} \cdot \frac{1}{1 + L_s} \cdot \left| \int_{\Omega} f_r(t) \cdot L_s \cdot \left( v + Jx^N(t) - (w + (1 - \lambda) Jx^N(t) \right) dt \right| \tag{3.39}
\leq \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s+1}} \cdot \frac{L_s}{1 + L_s} \cdot \| f_r \|_{L^1(\Omega,R)} \cdot \varepsilon \leq \varepsilon. \tag{3.40}
\]
Passing now to a subsequence \( \{\delta_{w+(1-\lambda)Jx^N(t)}\} \), which converges to a generalized gradient control \( \mu \in \mathcal{G}(K) \), we arrive at
\[
\lim_{N' \to \infty} g(\{\delta_{v+Jx^{N'}(t)}\}, \{\delta_{w+(1-\lambda)Jx^{N'}(t)}\}) = g(\{\nu'\}, \mu) \leq \varepsilon. \tag{3.41}
\]
From Theorem 2.18., 1) it follows that
\[
\sigma(\nu', A(\mu)) \leq g(\{\nu'\}, \mu) \leq \varepsilon \tag{3.42}
\]
where \( A(\mu) = \nu'' \) belongs to \( \text{S}^*(w) \). Exchanging the roles of \( v, w \in \text{int}(K) \) and assuming that
\[
|v - w| \leq \frac{\varepsilon}{4C_K} \cdot \min \left(1, \text{Dist}(w, \partial K)\right) \tag{3.43}
\]
we find, conversely, for arbitrary \( \nu'' \in \text{S}^*(w) \) some \( \nu' \in \text{S}^*(v) \) with \( \sigma(\nu', \nu'') \leq \varepsilon \). The proof is complete. \( \blacksquare \)

**Proof of Theorem 3.6.** The assertion follows in complete analogy to \([\text{WAGNER 07A}]\), p. 19, Proof of Theorem 3.6., 1), from Theorem 3.5., Definition 5.6. and Theorem 5.7. \( \blacksquare \)

b) **Semicontinuous extension \( S^b \) of the set-valued map \( S^* | \text{int}(K) \) to \( \partial K \).**

In a second step, we extend the set-valued map \( S^* \) to the boundary of \( K \). For this purpose, we will show that, along every ray \( R \) starting from the origin, the Painlevé-Kuratowski limit \( \lim_{K \to v_0, \ v \in R \cap \text{int}(K)} S^*(v) \) in the point \( v_0 \in R \cap \partial K \) exists. We start with
Definition 3.8. (S* as extension of S* to the boundary ∂K) We define the set-valued map $S^# : K \to \mathcal{P}(C(K))$ by

$$S^#(v_0) = \begin{cases} 
\lim_{v \to v_0, v \in R \cap \text{int}(K)} S^*(v_0) & | v_0 \in \text{int}(K); \\
S^*(v) & | v_0 \in \partial K.
\end{cases} \quad (3.44)$$

This definition will be justified by the following Theorem 3.9. $S^#$ is a set-valued map with nonempty, convex, weak*-sequentially compact images, and we obtain a representation of $S^#(v_0)$ for $v_0 \in \partial K$ in analogy to Definition 3.1. Finally, we prove that the set-valued map $S^#$ is upper semicontinuous (Theorem 3.12.).

Theorem 3.9.\(^{47}\) 1) (ε-δ relation for $f^*$ along rays starting from the origin) Assume that two points $v, w \in \text{int}(K)$ admit the following properties: a) $v, w$ are situated on the same ray $R$ starting from $\partial \sigma$, and b) $0 < \text{Dist}(w, \partial K) < \text{Dist}(v, \partial K) < \frac{1}{2}c_K$. Then $S^*$ obeys the following ε-δ estimate, which holds uniformly for all rays $R$ starting from $\partial \sigma$:

$$\text{Dist}(w, v) \leq \delta_2(\epsilon) \quad \Rightarrow \quad \text{for every } \nu'' \in S^*(w) \text{ there exists } \nu' \in S^*(v) \text{ with } \sigma(\nu'', \nu') \leq \epsilon \quad (3.45)$$

with $\delta_2(\epsilon) = \frac{1}{6} \delta(\epsilon) \cdot c_K/C_K$ where $c_K$ and $C_K$ are the quantities defined in the beginning of the section.

2) (Justification of Definition 3.8.) Along every ray $R$ starting from the origin, the following Painlevé-Kuratowski limit in the point $v_0 \in R \cap \partial K$ exists:

$$\lim_{v \to v_0, v \in R \cap \text{int}(K)} S^*(v). \quad (3.66)$$

3) (ε-δ relation for $S^#$ along rays starting from the origin) Under the assumptions of Part 1), we consider two points $v, w \in K$, which a) are situated on the same ray $R$ starting from $\partial \sigma$ and b) satisfy $0 \leq \text{Dist}(w, \partial K) \leq \text{Dist}(v, \partial K) < \frac{1}{2}c_K$. Then the ε-δ estimate from Part 1) can be extended to $S^#$:

$$\text{Dist}(w, v) \leq \delta_2(\epsilon) \quad \Rightarrow \quad \text{for every } \nu'' \in S^#(w) \text{ there exists } \nu' \in S^#(v) \text{ with } \sigma(\nu'', \nu') \leq \epsilon, \quad (3.47)$$

and again the estimate holds uniformly for all rays $R$ starting from $\partial \sigma$.

Theorem 3.10. (Properties of the sets $S^#(w)$ for $w \in \partial K$)

1) For every $w \in \partial K$, the set $S^#(w) \subseteq C(K)$ is nonempty, convex and weak*-sequentially compact.

2) For every $w \in \partial K$, the set $S^#(w)$ may be represented as

$$S^#(w) = \{ \nu \in C(K) \mid \text{there exist sequences } \{w_N\}, \text{int}(K) \text{ and } \{x_N\}, W_0^{1,\infty}(\Omega, \mathbb{R}^n) \text{ with }$$

$$\begin{align*}
a) & \lim_{N \to \infty} w_N = w, \\
b) & \lim_{N \to \infty} \|x_N\|_{C^0(\Omega, \mathbb{R}^n)} = 0, \\
c) & w_N + Jx_N(t) \in K \forall t \in \Omega \forall N \in \mathbb{N}, \\
d) & \{w_N + Jx_N\} \text{ generates the constant generalized gradient control } \nu = \{\nu\}.
\end{align*} \quad (3.48)$$

The proof of the upper semicontinuity of the set-valued map $S^#$ is based on the following assertion:

Theorem 3.11. (ε-δ relation for $S^#$ in points $v \in \partial K$)\(^{48}\) Let a point $v \in \partial K$ be given. Then for arbitrary $\epsilon > 0$ there exists $\delta_3(\epsilon, v) > 0$ with

---

\(^{47}\) Cf. [WAGNER 07a], p. 22 f., Theorem 3.12.

\(^{48}\) Cf. ibid., p. 23, Theorem 3.15.
Dist \((w, v) \leq \delta_4(\varepsilon, v) \implies \) for every \(\nu_w \in S^w(\nu)\) there exists \(\nu_v \in S^v(\nu)\) with \(\sigma(\nu_v, \nu_w) \leq 3\varepsilon\) (3.49) for all \(w \in K\).

**Theorem 3.12. (Upper semicontinuity of the set-valued map \(S^\#\))**

1) The set-valued map \(S^\#\) is upper semicontinuous on \(K\).

2) For all \(v_0 \in K\), it holds that \(S^\#(v_0) = \limsup_{v \to v_0, v \in \text{int}(K)} S^*(v)\).

Finally, we state

**Theorem 3.13. (\(S^\#\) in extremal points of \(K\))**

For every \(w \in \text{ext}(K)\), the set \(S^\#(w) = \{\delta_w\}\) is a singleton.

**Proof of Theorem 3.9.**

1) Let \(\text{Dist}(\varrho, v) = D\) and \(\text{Dist}(\varrho, w) = D + d\). Then it follows:

\[
0 < \frac{cK}{2} \leq D < D + d < C_K \implies \frac{cK}{2C_K} < \frac{cK}{2(D + d)} \leq \frac{D}{D + d} < 1,
\]

(3.50)

and the points \(v\) and \(w\) can be written as

\[
v = \frac{D}{D + d} w \quad \text{resp.} \quad w = \frac{D + d}{D} v.
\]

(3.51)

Choose now \(\varepsilon > 0\) and arbitrary \(\nu'' \in S^*(w)\). Then, in relation to \(\nu''\), there exists a sequence \(\{x^N\}, W^1_{0, \infty}(\Omega, \mathbb{R}^3)\), which possesses together with the constant sequence \(\{w\}\) the properties a) – d) from Definition 3.1. (cf. Lemma 3.3.). Consequently, we have for all \(N \in \mathbb{N}\):

\[
\frac{D + d}{D} v + Jx^N(t) \in K \quad (\forall) t \in \Omega \quad \text{resp.} \quad v + \frac{D}{D + d} Jx^N(t) \in \frac{D}{D + d} K \subset K \quad (\forall) t \in \Omega
\]

(3.52)

and

\[
\left| (w + Jx^N(t)) - (v + \frac{D}{D + d} Jx^N(t)) \right| \leq \frac{d}{D} \left| \frac{v}{D} + \frac{Jx^N(t)}{D + d} \right| \leq \frac{d}{D} \left| v + \frac{D}{D + d} Jx^N(t) \right| \leq d \cdot \frac{6C_K}{c_K}.
\]

(3.53)

With (3.53), we obtain

\[
g(\{\delta_{w + Jx^N(t)}\}, \{\delta_{v + \frac{D}{D + d} Jx^N(t)}\})
\]

\[
= \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s}} \cdot \frac{1}{1 + L_s} \cdot \left| \int_{\Omega} f_r(t) \left( g_s(w + Jx^N(t)) - g_s(v + \frac{D}{D + d} Jx^N(t)) \right) dt \right|
\]

\[
\leq \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s}} \cdot \frac{1}{1 + L_s} \cdot \int_{\Omega} \left| f_r(t) \cdot L_s \cdot \left| (w + Jx^N(t)) - (v + \frac{D}{D + d} Jx^N(t)) \right| dt \right|
\]

\[
\leq \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \frac{1}{2^{r+s}} \cdot \frac{1}{1 + L_s} \cdot \left\| f_r \right\|_{L_1(\Omega)} \cdot \frac{6C_K}{c_K} \cdot d;
\]

(3.54)

(3.55)

(3.56)

consequently, the implication

\[
|w - v| = d \leq \delta_2(\varepsilon) = \frac{\varepsilon}{6} \cdot \frac{cK}{C_K} \implies g(\{\delta_{w + Jx^N(t)}\}, \{\delta_{v + \frac{D}{D + d} Jx^N(t)}\}) \leq \varepsilon \quad \forall N \in \mathbb{N}
\]

(3.57)

49) Cf. [Wagner 07a], p. 23, Theorem 3.16.

50) Cf. ibid., p. 23, Theorem 3.14., 2).
Summing up, we arrive at

\[ \lim_{N \to \infty} \theta\left( \left\{ \delta_{w+N(x)} \right\}, \left\{ \delta_{v+N(x)} \right\} \right) = \theta\left( \left\{ \nu'' \right\}, \mu \right) \leq \varepsilon. \]  

(3.58)

By Theorem 2.18., 1), it holds that

\[ \sigma(\nu'', A(\mu)) \leq \theta\left( \left\{ \nu'' \right\}, \mu \right) \leq \varepsilon \]  

(3.59)

where \( A(\mu) = \nu' \) belongs to \( S^*(\nu) \).

2) Assume on the contrary that there exists some element

\[ \nu'' \in \left( \limsup_{v \to v_0}^{K} S^*(v) \right) \setminus \left( \liminf_{v \to v_0}^{K} S^*(v) \right). \]  

(3.60)

Then, by Definition 5.3., there exist sequences of points \( \left\{ w^N \right\}, R \cap \text{int}(K) \to v_0 \) and measures \( \left\{ \nu''_N \right\}, rca^{pr}(K) \) with \( \nu'' \in S(w^N) \cap \mathbb{N} \in \mathbb{N} \) and \( \lim_{N \to \infty} \sigma(\nu''_N, \nu'') = 0 \), but at the same time there exists another sequence of points \( \left\{ v^N \right\}, R \cap \text{int}(K) \to v_0 \) such that for any sequence of measures \( \left\{ \nu'_N \right\}, rca^{pr}(K) \) with \( \nu'_N \in S(v^N) \cap \mathbb{N} \in \mathbb{N} \) and \( \lim_{N \to \infty} \sigma(\nu'_N, \nu') = 0 \), the limits do not coincide: \( \nu'' \neq \nu' \). By a passage to suitable subsequences (without change of indices), we may guarantee that for all indices \( N \), it holds at the same time that

\[ |w^N - v_0| \leq \frac{1}{N} \cdot \frac{c_K}{12C_K}, \quad |v^N - v_0| \leq \frac{1}{N} \cdot \frac{c_K}{12C_K} \text{ and } |w^N - v_0| < |v^N - v_0| \]  

(3.61)

and thus

\[ |w^N - v^N| \leq \frac{1}{N} \cdot \frac{c_K}{6C_K} \]  

(3.62)

as well. By Part 1), for every \( \nu''_N \in S(w^N) \) there exists \( \nu'_N \in S^*(v^N) \) with \( \sigma(\nu''_N, \nu'_N) \leq 1/N \). The sequence \( \left\{ \nu'_N \right\} \) admits a weak*–convergent subsequence (index \( M \)); when passing to this subsequence, we may assume

\[ \sigma(\nu''_M, \nu'') \leq \frac{1}{M} \quad \text{and} \quad \sigma(\nu'_M, \nu') \leq \frac{1}{M} \]  

(3.63)

as well. We arrive now at a contradiction by the limit passage \( M \to \infty \) within the inequality

\[ 0 < \sigma(\nu'', \nu') \leq \sigma(\nu'', \nu''_M) + \sigma(\nu''_M, \nu'_M) + \sigma(\nu'_M, \nu') \leq \frac{1}{M} + \frac{1}{M} + \frac{1}{M}. \]  

(3.64)

3) Choose \( \varepsilon > 0 \). In view of Part 1), it remains to prove that the relation holds in the case where \( w \neq v \) with \( \text{Dist}(w, v) \leq \delta_2(\varepsilon) \) belongs to \( \partial K \). Then there exists a sequence of points \( \left\{ w^N \right\}, R \cap \text{int}(K) \to w \) with \( \text{Dist}(w^N, \partial K) < \text{Dist}(v, \partial K) \) and, consequently, \( \text{Dist}(w^N, v) \leq \delta_2(\varepsilon) \) for all \( N \in \mathbb{N} \). Let \( \nu'' \in S^#(w) \) be given. By Part 2), we find measures \( \nu''_N \in S^#(w^N) \) with \( \sigma(\nu'', \nu''_N) \leq \varepsilon^N \) and \( \left\{ \varepsilon^N \right\} \to 0 \). Furthermore, by Part 1), for every \( \nu''_N \in S^#(w^N) \) there exists \( \nu'_N \in S^#(v) \) with \( \sigma(\nu'_N, \nu''_N) \leq \varepsilon \). Since \( S^#(v) \subseteq rca^{pr}(K) \) is weak*-sequentially compact, the sequence \( \left\{ \nu''_N \right\}, S^#(v) \) admits a weak*-convergent subsequence with limit \( \nu' \in S^#(v) \) (we keep the index \( N \)). We may further assume that \( \sigma(\nu'_N, \nu') \leq \varepsilon^N \).

Summing up, we arrive at

\[ \sigma(\nu'', \nu') \leq \sigma(\nu'', \nu''_N) + \sigma(\nu''_N, \nu'_N) + \sigma(\nu'_N, \nu') \leq \varepsilon^N + \varepsilon + \varepsilon^N, \]  

(3.65)
what proves assertion 3) since \( \{ \varepsilon^N \} \to 0 \).

**Proof of Theorem 3.10.** 1) By Theorem 3.4., \( \delta_v \) belongs to \( S^*(v) \) for all \( v \in R \cap \text{int} (K) \), and for all sequences \( \{ w^N \} \), \( R \cap \text{int} (K) \to v_0 \), it holds that \( \lim_{N \to \infty} \sigma(\delta_{v_0}, \delta_{v^N}) = 0 \). Consequently, \( \delta_{v_0} \) belongs to \( S^#(v_0) = \lim_{v=v_0} \sup_{v \in R \cap \text{int} (K)} S^#(v) \), and this set is nonempty. As a Painlevé-Kuratowski limit, it is closed with respect to the topology generated by \( \sigma \) as well (Lemma 5.4., 2)). The convexity follows from Theorem 5.5., 2), the compactness again from Theorem 2.14., 2).

2) By Definition 3.8., it holds that

\[
S^#(w) = \lim_{v \to w, v \in R \cap \text{int} (K)} S^#(v) = \lim_{v \to w, v \in R \cap \text{int} (K)} \sup_{v \in R \cap \text{int} (K)} S^#(v)
\]

(3.66)

Consider now \( \nu \in S^#(w) \) together with sequences \( \{ w^N \} \), \( \text{int} (K) \to w \) and \( \{ \nu^N \} \), \( \text{rec}^p (K) \) with \( \nu^N \in S^#(w^N) \) for all \( N \in \mathbb{N} \) and \( \lim_{N \to \infty} \sigma(\nu^N, \nu) = 0 \). By Lemma 3.2. and 3.3., for every \( \nu^N \in S^#(w^N) \) there exist the constant sequence \( \{ w^N \} \) and a sequence \( \{ x^{N,M} \} \in W_0^{1,\infty} (\Omega, \mathbb{R}^n) \) with the properties a) – d) from Definition 3.1. We select a diagonal sequence \( \{ x^{N,M(N)} \} \) with

\[
g(\{ \delta_{w^N+Jx^{N,M(N)}(t)} \}, \{ \nu^N \}) \leq \frac{1}{N}.
\]

(3.67)

At the same time, we may assume that the sequence of generalized controls \( \{ \delta_{w^N+Jx^{N,M(N)}(t)} \} \) converges itself to a generalized gradient control \( \mu \in \mathcal{G}(K) \). Consequently, it holds that

\[
g(\{ \delta_{w^N+Jx^{N,M(N)}(t)} \}, \mu) \leq \frac{1}{N}.
\]

(3.68)

Applying the mean value theorem (Theorem 2.16.) to \( \mu \), we find that \( A(\mu) \) is generated by sequences with the properties a) – d). Finally, from Theorem 2.18., 1) it follows that

\[
\sigma(\nu, A(\mu)) \leq \sigma(\nu, \nu^N) + \sigma(\nu^N, A(\mu))
\]

\[
\leq \sigma(\nu, \nu^N) + g(\{ \nu^N \}, \{ \delta_{w^N+Jx^{N,M(N)}(t)} \} + g(\{ \delta_{w^N+Jx^{N,M(N)}(t)} \}, \mu) \to 0,
\]

(3.69)

what proves \( A(\mu) = \nu \). Thus \( \nu \in S^#(w) \) may be represented in the claimed way.

**Proof of Theorem 3.11.** Let us fix \( \varepsilon > 0 \). By Theorem 3.9., 2), there exists a point \( v' \in R_{v_0} \cap \text{int} (K) \) on the ray \( R_v = 0 \ v \) with

\[
0 < \text{Dist} (v', \partial K) \leq \min \left( 1, \frac{\delta_2(\varepsilon)}{2}, \frac{c_K}{2} \right) \quad \text{and} \quad \mathcal{H}(S^#(v), S^#(v')) \leq \varepsilon.
\]

(3.70)

From Theorem 3.5., we take \( \delta_1(\varepsilon) = \frac{1}{4} \varepsilon/C_K < 1 \). Defining \( \delta_3(\varepsilon, v) = \text{Dist} (v', \partial K) \), we infer from the Lipschitz continuity of the distance function \( \text{Dist} (\cdot, \partial K) \) (see [CLARKE 90], p. 50) that the points \( w' \in K(v', \frac{1}{2} \delta_1(\varepsilon) \delta_3(\varepsilon, v)) \) obey

\[
| \text{Dist} (w', \partial K) - \text{Dist} (v', \partial K) | \leq | w' - v' | \leq \frac{1}{2} \delta_1(\varepsilon) \delta_3(\varepsilon, v) \implies - \frac{1}{2} \delta_1(\varepsilon) \delta_3(\varepsilon, v) \leq \text{Dist} (w', \partial K) - \text{Dist} (v', \partial K) \implies - \frac{1}{2} \delta_1(\varepsilon) \delta_3(\varepsilon, v) + \text{Dist} (v', \partial K) = \delta_3(\varepsilon, v) \left( 1 - \frac{\delta_1(\varepsilon)}{2} \right) \leq \text{Dist} (w', \partial K).
\]

(3.71-3.73)
From $\delta_1(\varepsilon) < 1$ we conclude then
\[
\frac{\delta_3(\varepsilon, v)}{2} \leq \delta_3(\varepsilon, v) \left(1 - \frac{\delta_1(\varepsilon)}{2}\right) = \min \left\{1, \delta_3(\varepsilon, v), \delta_3(\varepsilon, v) \left(1 - \frac{\delta_1(\varepsilon)}{2}\right)\right\}
\]
(3.74)
\[
\leq \min \left\{1, \text{Dist}(v', \partial K), \text{Dist}(w', \partial K)\right\}.
\]

Summing up, we arrive at the implication
\[
|w' - v'| \leq \frac{1}{2} \delta_1(\varepsilon) \delta_3(\varepsilon, v) \quad \Rightarrow \quad |w' - v'| \leq \delta_1(\varepsilon) \cdot \min \left\{1, \text{Dist}(v', \partial K), \text{Dist}(w', \partial K)\right\}
\]
(3.75)
for arbitrary points $w' \in \text{int}(K)$, from which follows $\mathcal{H}(S^\#(v'), S^\#(w')) \leq \varepsilon$ (Theorem 3.5.). Consider now the points $w \in K$ with $|v - w| \leq \frac{1}{2} \delta_1(\varepsilon) \delta_3(\varepsilon, v) = \delta_4(\varepsilon, v)$. By the intercept theorems, for any of these points $w$ there exists a further point $w'' \in R_w \cap \text{int}(K)$ on the ray $R_w = \sigma w$ such that $w''$ belongs at the same time to $K(v', \frac{1}{2} \delta_1(\varepsilon) \delta_3(\varepsilon, v))$. For such a point $w''$, it holds that
\[
|w - w''| \leq |w - v| + |v - v'| + |v' - w''| \leq \frac{1}{2} \delta_1(\varepsilon) \delta_3(\varepsilon, v) + \delta_3(\varepsilon, v) + \frac{1}{2} \delta_1(\varepsilon) \delta_3(\varepsilon, v)
\]
(3.76)
\[
= \delta_3(\varepsilon, v) \left(1 + \delta_1(\varepsilon)\right) \leq 2 \delta_3(\varepsilon, v) \leq \delta_2(\varepsilon).
\]

Then by Theorem 3.9., 3), for every $\nu_w \in S^\#(w)$ there exists $\nu'' \in S^\#(w'')$ with $\sigma(\nu_w, \nu'') \leq \varepsilon$. Since $\mathcal{H}(S^\#(v'), S^\#(w'')) \leq \varepsilon$, in relation to $\nu''$ there exists $\nu' \in S^\#(v')$ with $\sigma(\nu', \nu'') \leq \varepsilon$, and since $\mathcal{H}(S^\#(v), S^\#(v')) \leq \varepsilon$, in relation to $\nu'$ there exists $\nu_v \in S^\#(v)$ with $\sigma(\nu_v, \nu') \leq \varepsilon$. Combining these inequalities, we find that for every $\nu_w \in S^\#(w)$ there exists $\nu_v \in S^\#(v)$ with
\[
\sigma(\nu_v, \nu_w) \leq \sigma(\nu_v, \nu') + \sigma(\nu', \nu'') + \sigma(\nu'', \nu_w) \leq 3 \varepsilon.
\]
(3.77)

**Proof of Theorem 3.12. 1)** It remains only to prove that $S^\#$ is upper semicontinuous in points $v_0 \in \partial K$.

From Theorem 3.11., for all $v \in K$ it follows:
\[
|v - v_0| \leq \frac{\varepsilon}{3}, v_0 \quad \Rightarrow \quad \text{to every } \nu \in S^\#(v) \text{ there exists } v_0 \in S^\#(v_0) \text{ with } \sigma(\nu, v_0) \leq \varepsilon.
\]
(3.78)

By Theorem 5.7., 2), this is equivalent with $\limsup_{v \to v_0} S^\#(v) \subseteq S^\#(v_0)$. According to Definition 5.6., 2), $S^\#$ is upper semicontinuous then in $v_0$.

**2)** Choose an arbitrary point $v_0 \in K$. Then we conclude from Definition 5.3. and Part 1):
\[
\limsup_{v \to v_0, v \in \text{int}(K)} S^*(v) = \limsup_{v \to v_0, v \in \text{int}(K)} S^\#(v) \subseteq \limsup_{v \to v_0, v \in K} S^\#(v) \subseteq S^\#(v_0).
\]
(3.79)

Conversely, if $R$ denotes the ray $\sigma w$ then it holds that
\[
S^\#(v_0) = \limsup_{v \to v_0, v \in R \cap \text{int}(K)} S^\#(v) = \limsup_{v \to v_0, v \in R \cap \text{int}(K)} S^\#(v)
\]
(3.80)
\[
= \{ \nu \in X \mid \exists \{v^N\}, R \cap \text{int}(K) \to v_0 \exists \{v^N\}, X \to \nu \text{ with } v^N \in S^\#(v^N) \forall N \in \mathbb{N} \}
\]
(3.81)
\[
\subseteq \{ \nu \in X \mid \exists \{v^N\}, \text{int}(K) \to v_0 \exists \{v^N\}, X \to \nu \text{ with } v^N \in S^\#(v^N) \forall N \in \mathbb{N} \}
\]
(3.82)
\[
= \limsup_{v \to v_0, v \in \text{int}(K)} S^\#(v),
\]
(3.83)
and the claimed equality results. ■
The proof of Theorem 3.13. will be postponed to the following section.

4. The representation theorem for \( f^{(qc)} \).

We arrive now at the desired representation of the lower semicontinuous quasiconvex envelope \( f^{(qc)} \) of \( f \in \mathcal{F}_K \) by means of the set-valued map \( S^\# \). Let us repeat our main result.

**Theorem 1.4. (Representation of \( f^{(qc)} \) in terms of probability measures)** Let a function \( f \in \mathcal{F}_K \) be given. Then for all \( w \in K \), \( f^{(qc)} \) admits the representation

\[
f^{(qc)}(w) = \min \left\{ \int_K f(v) \, d\nu(v) \mid \nu \in S^{(qc)}(w) \right\}
\]

where \( S^{(qc)}(w) = S^\#(w) \) with the set-valued map \( S^\#: K \to \mathcal{F}(\text{rca}_{pr}(K)) \) from Definition 3.8.

**Remark.** In view of Theorem 1.4., the notions \( S^\# \) and \( S^{(qc)} \) will be used as synonyms.

**Proof of Theorem 1.4.** Since the sets \( S^\#(w) \subseteq \text{rca}_{pr}(K) \) are weak*–sequentially compact, we may replace the minimum by infimum. We define

\[
h(w) = \inf \left\{ \int_K f(v) \, d\nu(v) \mid \nu \in S^\#(w) \right\}
\]

and distinguish the cases \( w \in \text{int}(K) \) and \( w \in \partial K \).

**Step 1.** Choose \( w \in \text{int}(K) \). Then by Theorem 1.3., \( f^{(qc)}(w) \) admits the representation

\[
f^{(qc)}(w) = f^*(w) = \inf \left\{ \frac{1}{|\Omega|} \int_\Omega f(v + Jx(t)) \, dt \mid x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n), v + Jx(t) \in K \ (\forall) t \in \Omega \right\}.
\]

Consequently, for every \( \varepsilon > 0 \) there exists \( x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n) \) with \( w + Jx(t) \in K \ (\forall) t \in \Omega \) and

\[
f^{(qc)}(w) \leq \frac{1}{|\Omega|} \int_\Omega f(w + Jx(t)) \, dt \leq f^{(qc)}(w) + \varepsilon.
\]

Applying the mean value theorem (Theorem 2.16.) to \( \mu = \{ \delta_{w+Jx(t)} \} \), we find a sequence \( \{ x^N \} \) of \( W_0^{1,\infty}(\Omega, \mathbb{R}^n) \) with

\[
\frac{1}{|\Omega|} \int_\Omega f(w + Jx(t)) \, dt = \lim_{N \to \infty} \frac{1}{|\Omega|} \int_\Omega f(w + Jx^N(t)) \, dt = \int_K f(v) \, d\nu(v)
\]

where the sequences \( \{ w \} \), \( \text{int}(K) \) and \( \{ x^N \} \) of \( W_0^{1,\infty}(\Omega, \mathbb{R}^n) \) possess the properties a) – d) from Definition 3.1. Thus \( \nu \) belongs to \( S^\#(w) \). It follows that for every \( \varepsilon > 0 \) there exists \( \nu \in S^\#(w) \) with

\[
\int_K f(v) \, d\nu(v) \leq f^{(qc)}(w) + \varepsilon,
\]

which proves the inequality \( h(w) \leq f^{(qc)}(w) \). Conversely, for every \( \varepsilon > 0 \) there exists \( \nu \in S^\#(w) \) with

\[
h(w) \leq \int_K f(v) \, d\nu(v) \leq h(w) + \frac{\varepsilon}{2}.
\]

By Lemma 3.2. and 3.3., \( \{ \nu \} \) may be generated by sequences \( \{ w \} \), \( \text{int}(K) \) and \( \{ x^N \} \) of \( W_0^{1,\infty}(\Omega, \mathbb{R}^n) \) with the properties a) – d) from Definition 3.1.; in particular, there exists \( x^N \in W_0^{1,\infty}(\Omega, \mathbb{R}^n) \) with \( w + Jx^N(t) \in K \ (\forall) t \in \Omega \) and

\[
\left| \int_K f(v) \, d\nu(v) - \frac{1}{|\Omega|} \int_\Omega f(w + Jx^N(t)) \, dt \right| \leq \frac{\varepsilon}{2}.
\]
It follows that

\[
\frac{1}{|\Omega|} \int_{\Omega} f(w + Jx^N(t)) dt \leq h(w) + \frac{\varepsilon}{2} + \left( \frac{1}{|\Omega|} \int_{\Omega} f(w + Jx^N(t)) dt \right) - \int_{K} f(v) d\nu(v) \leq h(w) + \varepsilon. \tag{4.9}
\]

We obtain

\[
f^{qc}(w) \leq \frac{1}{|\Omega|} \int_{\Omega} f(w + Jx^N(t)) dt \leq h(w) + \varepsilon, \tag{4.10}
\]

and the reverse inequality \( f^{qc}(w) \leq h(w) \) follows.

- **Step 2.** Let now \( w \in \partial K \) be given. Then by Theorem 1.3., we have for arbitrary sequences \( \{w^N\} \), \( R \cap \text{int}(K) \to w \) along the ray \( R = \emptyset w \)

\[
f^{qc}(w) = \lim_{N \to \infty} f^{qc}(w^N) = \lim_{N \to \infty} f^*(w^N). \tag{4.11}
\]

We fix \( \varepsilon > 0 \) and choose for every \( w^N \) a function \( x^N \in W^{1,\infty}_0(\Omega, \mathbb{R}^n) \) with \( w^N + Jx^N(t) \in K \) \( (\forall) t \in \Omega \) and

\[
f^{qc}(w^N) \leq \frac{1}{|\Omega|} \int_{\Omega} f(w^N + Jx^N(t)) dt \leq f^{qc}(w^N) + \varepsilon. \tag{4.12}
\]

The sequence of generalized controls \( \{\delta_{w^N + Jx^N(t)}\} \in \mathcal{G}(K) \) admits by Theorem 2.14., 2) a subsequence, which converges to a generalized gradient control \( \mu \in \mathcal{G}(K) \) (we keep the index \( N \)). It follows that

\[
f^{qc}(w) = \lim_{N \to \infty} f^{qc}(w^N) \leq \lim_{N \to \infty} \frac{1}{|\Omega|} \int_{\Omega} f(w^N + Jx^N(t)) dt = \frac{1}{|\Omega|} \int_{\Omega} \int_{K} f(v) d\mu(v) dt \leq \lim_{N \to \infty} f^{qc}(w^N) + \varepsilon = f^{qc}(w) + \varepsilon. \tag{4.13}
\]

Applying the mean value theorem (Theorem 2.16.) to \( \mu \), we find a sequence \( \{\tilde{x}^N\}, W^{1,\infty}_0(\Omega, \mathbb{R}^n) \) with

\[
\frac{1}{|\Omega|} \int_{\Omega} \int_{K} f(v) d\mu(v) dt = \lim_{N \to \infty} \frac{1}{|\Omega|} \int_{\Omega} f(w^N + J\tilde{x}^N(t)) dt = \int_{K} f(v) d\nu(v). \tag{4.14}
\]

By Theorem 3.10., 2), \( \nu \) belongs to \( S^#(w) \). Consequently, for every \( \varepsilon > 0 \) there exists \( \nu \in S^#(w) \) with

\[
\int_{K} f(v) d\nu(v) \leq f^{qc}(w) + \varepsilon, \tag{4.15}
\]

and we get the inequality \( h(w) \leq f^{qc}(w) \) as in Step 1. Conversely, for every \( \varepsilon > 0 \) there exists \( \nu \in S^#(w) \) with

\[
h(w) \leq \int_{K} f(v) d\nu(v) \leq h(w) + \frac{\varepsilon}{4}. \tag{4.16}
\]

Then by Definitions 5.3. and 3.8., there exist sequences \( \{w^N\}, R \cap \text{int}(K) \to w \) and \( \{\nu^N\} \), \( rca^{pr}(K) \) with \( \nu^N \in S^#(w^N) \) and \( \sigma(\nu^N, \nu) \to 0 \). Now we may choose an index \( N \), a probability measure \( \nu^N \in S^#(w^N) \) and a function \( x^N \in W^{1,\infty}_0(\Omega, \mathbb{R}^n) \) such that at the same time

\[
|f^{qc}(w^N) - f^{qc}(w)| \leq \frac{\varepsilon}{4}; \tag{4.17}
\]

\[
\left| \int_{K} f(v) \left( dv^N(v) - d\nu(v) \right) \right| \leq \frac{\varepsilon}{4}; \tag{4.18}
\]

\[
\left| \int_{K} f(v) dv^N(v) - \frac{1}{|\Omega|} \int_{\Omega} f(w^N + Jx^N(t)) dt \right| \leq \frac{\varepsilon}{4}. \tag{4.19}
\]
and \( w^N + Jx^N(t) \in K (\forall) t \in \Omega \) hold. It follows that
\[
h(w) \geq \int_K f(v) \, dv(v) - \frac{\varepsilon}{4} = \frac{1}{|\Omega|} \int_{\Omega} f(w^N + Jx^N(t)) \, dt
\]
\[
- \left( \frac{1}{|\Omega|} \int_{\Omega} f(w^N + Jx^N(t)) \, dt - \int_K f(v) \, dv(v) \right) - \left( \int_K f(v) \, dv(v) - \int_K f(v) \, dv(v) \right) - \frac{\varepsilon}{4}
\]
\[
\geq f^{(\nu)}(w^N) - |...| - ... - |...| - \frac{\varepsilon}{4}
\]
\[
= f^{(\nu)}(w) - (f^{(\nu)}(w) - f^{(\nu)}(w^N)) - \frac{3\varepsilon}{4} \geq f^{(\nu)}(w) - |...| - \frac{3\varepsilon}{4} \geq f^{(\nu)}(w) - \varepsilon.
\]
We arrive at \( f^{(\nu)}(w) \leq h(w) \), and the proof is complete. ■

**Proof of Theorem 3.13.** Let \( w \in \text{ext}(K) \) be given. Then \( \delta_w \in S^\#(w) \) by Lemma 3.3. and Proof of Theorem 3.10., 1). Let now arbitrary \( \nu \in S^\#(w) \) be given. By [Wagner 07a], p. 23, Theorem 3.14., 2), and the Theorems 1.3. and 1.4. above, we have
\[
f^{(\nu)}(w) = f(w) = \int_K f(v) \, dv(v) \ \forall f \in \mathcal{F}_K.
\]
Since \( \mathcal{F}_K \) and \( C^0(K, \mathbb{R}) \) are isomorphical, \( \nu \) is uniquely determined by this variational equality as a linear, continuous functional on \( C^0(K, \mathbb{R}) \), and we arrive at \( \nu = \delta_w \). ■

5. Appendix: Set-valued maps.

a) Painlevé-Kuratowski limits.

**Definition 5.1. (Painlevé-Kuratowski limits for set sequences)** [Aubin/Frankowska 90], p. 17, Definition 1.1.1.; see also [Rockafellar/Wets 98], p. 109, Definition 4.1. In [Rockafellar/Wets 98], all definitions and theorems have been formulated within the framework of the euclidean space \( \mathbb{R}^n \) only. However, numerous assertions presented there remain valid within arbitrary metric spaces.

**Definition 5.2. (Hausdorff distance in the metric space \( X \))** [Rockafellar/Wets 98], p. 117, Example 4.13.
Definition 5.3. (Painlevé-Kuratowski limits for set-valued maps) Let a nonempty, compact subset $K \subset \mathbb{R}^{mn}$ with $0 \in \text{int}(K)$ and a compact metric space $[X, \sigma]$ be given. We consider a set-valued map $S: K \to \mathcal{P}(X)$ with nonempty, closed images, and define for $v_0 \in K$

\[
\liminf_{v \to v_0}^K S(v) = \bigcap_{N \to \infty} \liminf_{v^N \to v_0}^K S(v^N) \quad \text{(5.5)}
\]

\[
\limsup_{v \to v_0}^K S(v) = \bigcup_{N \to \infty} \limsup_{v^N \to v_0}^K S(v^N) \quad \text{(5.6)}
\]

\[
\lim^K S(v) = E \iff \liminf_{v \to v_0}^K S(v) = \limsup_{v \to v_0}^K S(v) = E. \quad \text{(5.7)}
\]

Lemma 5.4. (Closedness of the Painlevé-Kuratowski limits) Assume that $K \subset \mathbb{R}^{mn}$ is nonempty and compact with $0 \in \text{int}(K)$ and $[X, \sigma]$ is a compact metric space.

1) For every set sequence $\{E_N\}, \mathcal{P}(X)$, the sets $\liminf_{N \to \infty}^K E_N, \limsup_{N \to \infty}^K E_N$ and (in the case of its existence) $\lim^K_{N \to \infty} E_N$ are closed with respect to the topology generated by $\sigma$.

2) Assume that $S: K \to \mathcal{P}(X)$ is a set-valued map with nonempty, closed images. Then for all $v_0 \in K$, the sets $\liminf_{v \to v_0}^K S(v), \limsup_{v \to v_0}^K S(v)$ and (in case of its existence) $\lim^K_{v \to v_0} S(v)$ are closed with respect to the topology generated by $\sigma$.

Theorem 5.5. (Convexity of the Painlevé-Kuratowski limits inferior) Consider a nonempty, compact set $K \subset \mathbb{R}^{mn}$ with $0 \in \text{int}(K)$ and a linear topological space, which contains $X$ as convex and sequentially compact subset. Assume further that the restriction of the topology to $X$ is metrizable, and thus $[X, \sigma]$ forms a compact metric space.

1) If $\{E_N\}, \mathcal{P}(X)$ is a sequence of convex sets then the sets $\liminf_{N \to \infty}^K E_N$ and (in case of its existence) $\lim^K_{N \to \infty} E_N$ are convex as well.

2) If $S: K \to \mathcal{P}(X)$ is a set-valued map with nonempty, closed, convex images then for all $v_0 \in K$, the sets $\liminf_{v \to v_0}^K S(v)$ and (in case of its existence) $\lim^K_{v \to v_0} S(v)$ are convex as well.

The assumptions of Theorem 5.5. are particularly satisfied for $X = \text{rca}^{pr}(K)$, endowed with the metric $\sigma$ from Definition 2.1. In fact, by Lemma 2.2., the restriction of the weak* topology of the space $\text{rca}^{pr}(K)$ to its (norm-) closed unit ball can be metrized by $\sigma$; consequently, the operations of addition and scalar multiplication are continuous with respect to this metric, and $\text{rca}^{pr}(K)$ forms a convex, weak*-sequentially compact subset of the unit ball.

Proof of Lemma 5.4. 2) For $\liminf_{v \to v_0}^K S(v)$ and (in case of its existence) $\lim^K_{v \to v_0} S(v)$, the assertion follows from Part 1) together with the representations of the limits according to Definition 5.3. Consider now a sequence $\{v^K\}, \limsup^K_{v \to v_0} S(v) \to \nu \in X$. Then for every index $K$ there exist sequences $\{v^{N,K}\}, K \to v_0$ and $\{v^{N,K}\}, X \to \nu^K$ with $\nu^{N,K} \in S(v^{N,K}) \forall N \in \mathbb{N}$, and for every $\varepsilon > 0$ there exists an index $K(\varepsilon)$ with $\sigma(\nu^K, \nu) \leq \varepsilon$ as well as an index $N(\varepsilon)$ with $\sigma(\nu^{N,K}, (K(\varepsilon))) \leq \varepsilon$ and $|v^{N,K}(\varepsilon) - v_0| \leq \varepsilon$. Consequently, there exist sequences $\{v^{M}\}, K \to v_0$ and $\{v^{M}\}, X \to \nu$ with $\nu^M \in S(v^M) \forall M \in \mathbb{N}$, and $\nu$ belongs to $\limsup^K_{v \to v_0} S(v)$ as well. ■

\[\text{[Aubin/Frankowska 90], p. 41, Definition 1.4.6.; see also Rockafellar/Wets 98, p. 152.}\]

\[\text{[Rockafellar/Wets 98], p. 111, Proposition 4.4.}\]

\[\text{See ibid., p. 119, Proposition 4.15.}\]
Proof of Theorem 5.5. 1) The proof of [Rockafellar/Wets 98], p. 119, Proposition 4.15., can be immediately carried over to the present analytical situation.

2) The assertion follows from Part 1) together with the representation of \( \liminf_{v \to v_0} K(v) \) as an intersection (Definition 5.3.).

b) Semicontinuous and continuous set-valued maps.

Definition 5.6. (Semicontinuity and continuity of set-valued maps) 56) Let a nonempty, compact set \( K \subseteq \mathbb{R}^n \) with \( \sigma \in \text{int} (K) \) and a compact metric space \( [X, \sigma] \) be given. We consider a set-valued map \( S : K \to \mathcal{P}(X) \) with nonempty, closed images.

1) The set-valued map \( S \) is called lower semicontinuous in \( v_0 \in K \) if \( S(v_0) \subseteq \liminf_{v \to v_0} S(v) \) holds.

2) The set-valued map \( S \) is called upper semicontinuous in \( v_0 \in K \) if \( \limsup_{v \to v_0} S(v) \subseteq S(v_0) \) holds.

3) The set-valued map \( S \) is called continuous in \( v_0 \in K \) if \( S(v) = \lim_{v \to v_0} S(v) \) holds.

Theorem 5.7. (Conditions for semicontinuity and continuity of set-valued maps) 57) Let a nonempty, compact set \( K \subseteq \mathbb{R}^n \) with \( \sigma \in \text{int} (K) \) and a compact metric space \( [X, \sigma] \) be given. Assume further that \( S : K \to \mathcal{P}(X) \) is a set-valued map with nonempty, closed images, and \( E \subseteq X \) is a nonempty, closed subset of \( X \).

1) \( E \subseteq \liminf_{v \to v_0} S(v) \iff \forall \varepsilon > 0 \exists \delta (\varepsilon) > 0: \)

If \( |v - v_0| \leq \delta (\varepsilon) \) then there exists for every \( \nu \in E \) an element \( \nu_0 \in S(v) \) with \( \sigma(\nu, \nu_0) \leq \varepsilon \).

2) \( \limsup_{v \to v_0} S(v) \subseteq E \iff \forall \varepsilon > 0 \exists \delta (\varepsilon) > 0: \)

If \( |v - v_0| \leq \delta (\varepsilon) \) then there exists for every \( \nu_0 \in S(v) \) an element \( \nu \in E \) with \( \sigma(\nu, \nu_0) \leq \varepsilon \).

3) \( E = \lim_{v \to v_0} S(v) \iff \lim_{v \to v_0} \mathcal{H}(S(v), E) = 0 \).

References.


56) [Aubin/Frankowska 90], p. 39 f., Definitions 1.4.2. and 1.4.3.; see also [Rockafellar/Wets 98], p. 152, Definition 5.4.


22. [Lur’e 75] Lur’e, K. A. (Лурье, К. А.): *Оптимальное управление в задачах математической физики*. Наука; Москва 1975


26. [Rockafellar/Wets 98] Rockafellar, R. T.; Wets, R. J.-B.: *Variational Analysis*. Springer; Berlin etc. 1998 (Grundlehren 317)


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