Differentiability of the lower semicontinuous quasiconvex envelope

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1. Introduction.

a) The lower semicontinuous quasiconvex envelope.

The present paper is motivated by the study of multidimensional control problems of Dieudonné-Rashevsky type. In the simplest case, a problem of this type will be obtained when adding to the basic problem of multidimensional calculus of variations,

\[ F(x) = \int_\Omega r(t, x(t)) \, dt \rightarrow \inf \quad x \in W^{1,p}_0(\Omega, \mathbb{R}^n), \quad \Omega \subset \mathbb{R}^m, \]  

restrictions for the partial derivatives of \( x \), e. g.

\[ Jx(t) = \left( \begin{array}{ccc} \frac{\partial x_1}{\partial t_1}(t) & \ldots & \frac{\partial x_1}{\partial t_m}(t) \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1}(t) & \ldots & \frac{\partial x_n}{\partial t_m}(t) \end{array} \right) \in K \subset \mathbb{R}^{nm} \quad (\forall) \ t \in \Omega. \]  

Problems of this kind result from applications in elasticity theory, geometry and image processing, spanning from models for torsion of prismatic bars and optimization problems for convex bodies under geometrical restrictions to the problem of edge detection within the optical flow. In their papers on underdetermined boundary value problems for nonlinear first-order PDE’s, Dacorogna and Marcellini arrived at Dieudonné-Rashevsky type problems as well. All mentioned applications have in common that the gradient restriction (1.2) is related to a convex body \( K \) with \( o \in \text{int}(K) \), while the integrand \( r(t, \xi, v) \) in (1.1) is a possibly nonconvex function, whose natural range of definition is the subset \( \Omega \times \mathbb{R}^n \times K \) instead of the whole space.

In the relaxation theory of multidimensional variational problems, it is a well-known fact that the investigation of general integrands \( r(t, \xi, v) \) can be reduced to the special case where the integrand depends on \( v \) only. Consequently, as in the author’s previous papers concerned with existence and relaxation results for Dieudonné-Rashevsky type problems, we confine ourselves to the investigation of unbounded integrands \( f(v) \) and an appropriate semiconvex envelope, which turns out to be the lower semicontinuous quasiconvex envelope. Then we must specify the notion of quasiconvexity for functions with values in \( \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \) as follows:

\[ \text{References:} \]

01 \[ \text{Lur'e 75], pp. 240 ff., [Ting 69A], p. 531 f., [Ting 69B], [Wagner 96], pp. 76 ff.} \]

02 \[ \text{Andrejewa/Kl"{o}tzler 84A], [Andrejewa/Kl"{o}tzler 84B], p. 149 f.} \]

03 \[ \text{Brune 07], [Wagner 07B], pp. 12 ff, Section 4.} \]

04 \[ \text{Dacorogna/Marcellini 97], [Dacorogna/Marcellini 98], [Dacorogna/Marcellini 99].} \]

05 \[ \text{E. g. nonconvex regularization terms of Perona-Malik type, cf. [Aubert/Kornprobst 02], pp. 80 ff., and [Kawohl 04].} \]

06 \[ \text{Dacorogna 08], pp. 369 ff. and 416 ff.} \]

07 \[ \text{Wagner 07A] – [Wagner 07e].} \]
Definition 1.1. (Quasiconvex function with values in $\mathbb{R}$) A function $f : \mathbb{R}^{nm} \to \bar{\mathbb{R}}$ with the following properties is said to be quasiconvex:

1) dom $(f) \subseteq \mathbb{R}^{nm}$ is a (nonempty) Borel set;

2) $f \big{|} \text{dom} \,(f)$ is Borel measurable and bounded from below on every bounded subset of dom $(f)$;

3) for all $v \in \mathbb{R}^{nm}$, $f$ satisfies Morrey’s integral inequality:

$$f(v) \leq \frac{1}{|\Omega|} \int_{\Omega} f(v + Jx(t)) \, dt \quad \forall x \in W_{0}^{1,\infty}(\Omega, \mathbb{R}^{n});$$

or equivalently

$$f(v) = \inf \left\{ \frac{1}{|\Omega|} \int_{\Omega} f(v + Jx(t)) \, dt \big{|} x \in W_{0}^{1,\infty}(\Omega, \mathbb{R}^{n}) \right\}. \quad (1.3)$$

Here $\Omega \subset \mathbb{R}^{m}$ is the closure of a bounded strongly Lipschitz domain.

For the lower semicontinuous quasiconvex envelope of an unbounded function, we adopt the following definition:

Definition 1.2. (Lower semicontinuous quasiconvex envelope $f^{(qc)}$ for functions with values in $\mathbb{R}$) For any function $f : \mathbb{R}^{nm} \to \bar{\mathbb{R}}$ bounded from below, we define

$$f^{(qc)}(v) = \sup \left\{ g(v) \big{|} g : \mathbb{R}^{nm} \to \bar{\mathbb{R}} \text{ quasiconvex and lower semicontinuous}, \quad g(v) \leq f(v) \quad \forall v \in \mathbb{R}^{nm} \right\}. \quad (1.5)$$

Obviously, Definition 1.2. generalizes the usual formation of a quasiconvex envelope since the quasiconvex functions $g$ below a finite function $f$ are continuous from the outset.

b) Main result: Differentiability of $f^{(qc)}$.

In the present paper, we will study the differentiability properties of the lower semicontinuous quasiconvex envelope for integrands $f(v)$ within the following function class:

Definition 1.3. (Function class $F_{K}$) Let $K \subset \mathbb{R}^{nm}$ be a given convex body with $\sigma \in \text{int} \,(K)$. We say that a function $f : \mathbb{R}^{nm} \to \bar{\mathbb{R}}$ belongs to the class $F_{K}$ iff $K \in C^{0}(K, \mathbb{R})$ and $f \big{|} (\mathbb{R}^{nm} \setminus K) \equiv (+\infty)$.

For a given function $f \in F_{K}$, the convex envelope $f^{c}$ as well as the lower semicontinuous quasiconvex envelope $f^{(qc)}$ are rank one convex $^{11)}$ what implies their local Lipschitz continuity on int $(K)$. $^{12)}$ Consequently, by Rademacher’s theorem, $^{13)}$ both functions are continuously differentiable almost everywhere on int $(K)$. For the convex envelope, this assertion may be sharpened as follows:

$^{08)}$ [WAGNER 07A], p. 6, Definition 2.9., as specification of [BALL/MURAT 84], p. 228, Definition 2.1., in the case $p = (+\infty)$.

$^{09)}$ [WAGNER 07A], p. 9, Definition 2.14., 2).

$^{10)}$ The presence of a convex gradient restriction $Jx(t) \in K$ within Dieudonné-Rashevsky type problems is reflected here. The necessity to extend the integrand with $(+\infty)$ to $v \in \mathbb{R}^{nm} \setminus K$ before the forming of the envelope has been confirmed by the examples in [WAGNER 07A], pp. 11 ff., Section 2.d).

$^{11)}$ [DACOROGNA 08], p. 159, Theorem 5.3., (i), and Theorem 2.4. below.

$^{12)}$ See Theorem 2.2. below.

$^{13)}$ [EVANS/GARIEPY 92], p. 81, Theorem 2.
Theorem 1.4. (Differentiability of $f^c$ on int $(K)$) \[ \text{Assume that a function } f \in F_K \text{ is defined through} \]

$$f(v) = \begin{cases} \tilde{f}(v) & v \in K; \\ (+\infty) & v \in \mathbb{R}^{nm} \setminus K \end{cases}$$ \hspace{1cm} (1.6)$$

where $\tilde{f} : \mathbb{R}^{nm} \to \mathbb{R}$ is a continuous function, which is continuously differentiable for all $v \in K$ at least. Then the convex envelope $f^c$ is continuously differentiable for all $v \in \text{int} (K)$.

In the case of the quasiconvex envelope $f^{qc}$ of a finite function $f : \mathbb{R}^{nm} \to \mathbb{R}$ bounded from below, Ball/Kirchheim/Kristensen have been proved that the differentiability of $f$ together with some growth conditions implies the differentiability of $f^{qc}$. The main result of the present paper is the extension of this result to the lower semicontinuous quasiconvex envelope:

Theorem 1.5. (Differentiability of $f^{qc}$ for $f \in F_K$) \[ \text{Assume that a function } f \in F_K \text{ is defined through} \]

$$f(v) = \begin{cases} \tilde{f}(v) & v \in K; \\ (+\infty) & v \in \mathbb{R}^{nm} \setminus K \end{cases}$$ \hspace{1cm} (1.7)$$

where $\tilde{f} : \mathbb{R}^{nm} \to \mathbb{R}$ is a continuous function, which is continuously differentiable on some open neighbourhood of $K$ at least. Then the lower semicontinuous quasiconvex envelope $f^{qc}$ is continuously differentiable for all $v \in \text{int} (K)$.

As a corollary, we obtain the Lipschitz continuity of $f^{qc}$ on the whole $K$ in the special case $\partial K = \text{ext} (K)$.

Theorem 1.6. (Global Lipschitz continuity of $f^{qc}$ in the case $\partial K = \text{ext} (K)$) \[ \text{Assume that } K \text{ is a convex body with } \emptyset \in \text{int} (K) \text{ and } \partial K = \text{ext} (K), \text{ and that a function } f \in F_K \text{ satisfies the assumptions of Theorem 1.5. Then } f^{qc} \text{ is (globally) Lipschitz continuous on } K. \]

c) Outline of the paper.

We close this introduction with a synopsis of the notations and abbreviations to be used throughout the paper. In Section 2, we collect some ingredients for the proof of Theorem 1.5. In particular, we summarize the known facts about $f^{qc}$ and present a brief introduction to the theory of generalized controls (“Young measures”). Having all tools to our disposal, we turn in Section 3 to the proof of Theorems 1.5. and 1.6.

d) Notations and abbreviations.

Let $k \in \{0, 1, \ldots, \infty\}$ and $1 \leq p \leq \infty$. Then $C^k(\Omega, \mathbb{R}^r)$, $L^p(\Omega, \mathbb{R}^r)$ and $W^{k,p}(\Omega, \mathbb{R}^r)$ denote the spaces of $r$-dimensional vector functions whose components are $k$-times continuously differentiable, belong to $L^p(\Omega)$ or to the Sobolev space of $L^p(\Omega)$-functions with weak derivatives up to $k$th order in $L^p(\Omega)$ respectively. In addition, functions within the subspace $W^{1,\infty}(\Omega, \mathbb{R}^r) \subset W^{1,\infty}(\Omega, \mathbb{R}^r)$ admit a (Lipschitz-) continuous representative with zero boundary values. The Jacobi matrix of $x$ is abbreviated as $Jx$. Note that the function class $F_K$ from Definition 1.3. and the Banach space $C^0(K, \mathbb{R})$ are isomorphical and isometrical.

The space of Radon measures (signed regular measures) acting on the $\sigma$-algebra of the Borel subsets of a compact set $K \subset \mathbb{R}^{nm}$ is denoted by $rca (K)$. Endowed with the total variation norm $\text{tv} \mu (K)$, it forms a

\[ \text{[Griewank/Rabier 90], p. 701, Corollary 3.1.} \]
\[ \text{[Ball/Kirchheim/Kristensen 00], p. 334, Theorem A.} \]
\[ \text{[Evans/Gariepy 92], p. 131, Theorem 5.} \]
Banach space.\textsuperscript{17) Due to the compactness of K, the dual space \( (C^{0}(K, \mathbb{R}))^{\ast} \) and \( rca(\mathbb{K}) \) are isomorphical,\textsuperscript{18) consequently, every linear, continuous functional on \( C^{0}(K, \mathbb{R}) \) may be represented as an integral with respect to a Radon measure \( \nu \in rca(\mathbb{K}) \). The subset of the probability measures, equipped with the restriction of the weak* topology, will be denoted by \( rca^{pr}(\mathbb{K}) \).\textsuperscript{19) The Dirac measure concentrated in \( v \in \mathbb{K} \) is denoted by \( \delta_v \). If X is an arbitrary set then \( \Psi(X) \) denotes the set of all subsets of X.

We denote by int (A), \( \partial A \), cl (A), co (A) and \( |A| \) the interior, boundary, closure, the convex hull and the \( r \)-dimensional Lebesgue measure of the set \( A \subseteq \mathbb{R}^r \), respectively. If \( K \subseteq \mathbb{R}^{nm} \) is a convex set then a point \( v \in K \) is called extremal point of \( K \) iff from \( v = x'v' + \lambda''v'' \), \( x', \lambda' > 0, \lambda' + \lambda'' = 1, v', v'' \in K \) it follows that \( v' = v'' = v \). The set of all extremal points of \( K \) is denoted by \( \text{ext}(K) \). For a convex body, i.e. a compact, convex subset \( K \subseteq \mathbb{R}^{nm} \), \( \text{ext}(K) \) is always nonempty. Further, we set \( \mathbb{R} = \mathbb{R} \cup \{ (+\infty) \} \) and equip \( \mathbb{R} \) with the natural topological and order structures where \( (+\infty) \) is the greatest element. Throughout the whole paper, we consider only \( \text{proper functions} \ f: \mathbb{R}^{nm} \to \mathbb{R} \), assuming that \( \text{dom}(f) = \{ v \in \mathbb{R}^{nm} \mid f(v) < (+\infty) \} \) is always nonempty. The restriction of the function \( f \) to the subset \( A \) of its range of definition is denoted by \( f|A \).

We close this subsection with \textit{three nonstandard notations}. \{ \{ x^N \} : \Lambda \} denotes a sequence \{ \{ x^N \} \} with members \( x^N \in A \). If \( A \subseteq \mathbb{R}^r \) then the abbreviation \( (\forall) t \in A \) has to be read as “for almost all \( t \in A \)” resp. “for all \( t \in A \) except a \( r \)-dimensional Lebesgue null set”. The symbol \( \sigma \) denotes, depending on the context, the zero element resp. the zero function of the underlying space.

2. Ingredients for the proof.

a) Generalized notions of convexity.

We start with a recall of the generalized convexity notions to be used in the present paper.

\textbf{Definition 2.1.} 1) \( \text{(Rank one convex function)} \) A function \( f: \mathbb{R}^{nm} \to \mathbb{R} \) is said to be rank one convex if Jensen’s inequality is satisfied in any rank one direction: for every \( v', v'' \in \mathbb{R}^{nm} \) (considered as \( (n,m) \)-matrices) it holds that

\[
\text{Rank}(v' - v'') \leq 1 \implies f(\lambda' v' + \lambda'' v'') \leq \lambda' f(v') + \lambda'' f(v'') \quad \forall \lambda', \lambda'' \geq 0, \lambda' + \lambda'' = 1. \tag{2.1}
\]

2) \( \text{(Separately convex function)} \) A function \( f: \mathbb{R}^{nm} \to \mathbb{R} \) is said to be separately convex if it is convex in every variable \( v_{ij} \), while the other arguments are fixed.

For functions \( f: \mathbb{R}^{nm} \to \mathbb{R} \), rank one convexity implies separate convexity.\textsuperscript{20) The following theorem states that local Lipschitz continuity can be guaranteed even for separately convex functions.

\textbf{Theorem 2.2.} \( \text{(Lipschitz continuity of separately convex functions)} \)\textsuperscript{21) Any separately convex function \( f: \mathbb{R}^{nm} \to \mathbb{R} \) is locally Lipschitz continuous on \( \text{int}(\text{dom}(f)) \).}

\textsuperscript{17} [DUNFORD/SCHWARTZ 88], p. 161 f.

\textsuperscript{18} Ibid., p. 265, Theorem 3.

\textsuperscript{19} Cf. [WAGNER 07d], pp. 4 ff., Section 2.a). It has been proven there that \( rca^{pr}(\mathbb{K}) \) forms actually a separable, compact metric space.

\textsuperscript{20} [DACOROGNA 08], p. 159 f. Remark 5.4., (iii).

\textsuperscript{21} Ibid., p. 47, Theorem 2.31.
b) Semicontinuity, continuity and convexity properties of $f(q_e)$.

The following results have been obtained in [Wagner 07a].

**Theorem 2.3. (Semicontinuity and continuity of $f(q_e)$)**\(^{22}\) Let a function $f \in \mathcal{F}_K$ be given.

1) The function $f(q_e) : \mathbb{R}^{nm} \to \mathbb{R}$ is lower semicontinuous.

2) $f(q_e)$ is continuous in every point $v \in \text{int} \left( K \right)$.

3) The restriction $f(q_e) | K$ is continuous in every point $v \in \text{ext} \left( K \right)$, and there it holds that $f(q_e)(v) = f(v)$.

Consequently, if $\partial K = \text{ext} \left( K \right)$ then $f(q_e) | K$ is continuous on the whole set $K$, and $f(q_e)$ belongs to $\mathcal{F}_K$.

**Theorem 2.4. (Quasiconvexity and rank one convexity of $f(q_e)$)**\(^{23}\) Let a function $f \in \mathcal{F}_K$ be given.

Then the function $f(q_e) : \mathbb{R}^{nm} \to \mathbb{R}$ is quasiconvex (in the sense of Definition 1.1.) as well as rank one convex.

c) Generalized controls.

A measure-valued map $\mu : \Omega \to \text{rea}_p^r(K)$ with $t \mapsto \mu_t$ is called a generalized control (“Young measure”) if, for any continuous function $g \in C^0(K, \mathbb{R})$, the function $h(t) = \int_K g(v) \, d\mu_t(v)$ is Borel measurable on $\Omega$.\(^{24}\) Two generalized controls $\mu' = \{ \mu'_t \}$ and $\mu'' = \{ \mu''_t \}$ will be identified if $\mu'_t \equiv \mu''_t$ holds for almost all $t \in \Omega$. The set of all equivalence classes of generalized controls will be denoted by $\mathcal{Y}(K)$. The convergence of a sequence $\{ \mu^N \}$, $\mathcal{Y}(K)$ towards the limit $\mu \in \mathcal{Y}(K)$ is defined through

$$\mu^N \to \mu \iff \int_{\Omega} \int_K f(t) \, g(v) \left( d\mu^N_t(v) - d\mu_t(v) \right) dt \to 0 \quad \text{for all } f \in L^1(\Omega, \mathbb{R}), \, g \in C^0(K, \mathbb{R}). \quad (2.2)$$

**Definition 2.5. (Generating sequences for generalized controls)**\(^{25}\) We say that the sequence $\{ u^N \}$, $L^\infty(\Omega, \mathbb{R}^{nm})$ generates the generalized control $\mu \in \mathcal{Y}(K)$ if $u^N(t) \in K$ for all $t \in \Omega \cap N \in \mathbb{N}$ and

$$\lim_{N \to \infty} \int_{\Omega} f(t) \, g(u^N(t)) \, dt = \int_{\Omega} \int_K f(t) \, g(v) \, d\delta_{u^N(t)}(v) \, dt = \int_{\Omega} \int_K f(t) \, g(v) \, d\mu_t(v) \, dt, \quad (2.3)$$

for all $f \in L^1(\Omega, \mathbb{R}), \, g \in C^0(K, \mathbb{R})$.

**Definition 2.6. (Generalized gradient controls, “gradient Young measures”)**\(^{26}\) A measure-valued map $\mu \in \mathcal{Y}(K)$ is called a generalized gradient control if it is generated (in the sense of Definition 2.5.) by a sequence $\{ Jx^N \}$, $L^\infty(\Omega, \mathbb{R}^{nm})$ with $x \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ and $Jx^N(t) \in K$ for all $t \in \Omega \cap N \in \mathbb{N}$. The set of equivalence classes of generalized gradient controls will be denoted by $\mathcal{G}(K) \subseteq \mathcal{Y}(K)$.

**Theorem 2.7. (Properties of the spaces $\mathcal{Y}(K)$ and $\mathcal{G}(K)$)**

1)\(^{27}\) Every sequence $\{ u^N \}$, $L^\infty(\Omega, \mathbb{R}^{nm})$ with $u^N(t) \in K$ for all $t \in \Omega \cap N \in \mathbb{N}$ admits a weak* convergent subsequence, which generates a generalized control $\mu \in \mathcal{Y}(K)$.

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\(^{23}\) Ibid., p. 27, Theorem 3.19., and p. 29, Theorem 4.1.

\(^{24}\) Cf. [Gamkrelidze 78], pp. 23 ff., and [Müller 99], p. 115 ff.

\(^{25}\) Cf. [Pedregal 97], pp. 96 ff.

\(^{26}\) [Kinderlehrer/Pedregal 91], p. 333, [Müller 99], p. 126, Definition 4.1.

\(^{27}\) [Müller 99], p. 115 f., Theorem 3.1.
2) Every sequence \( \{ x^N \}, W^{1,\infty}(\Omega, \mathbb{R}^n) \) with \( \| x^N \|_{L^\infty(\Omega, \mathbb{R}^n)} \leq C \), \( Jx^N(t) \in K \) \( \forall t \in \Omega \) \( \forall N \in \mathbb{N} \) admits a subsequence \( \{ x^{N'} \} \) with \( x^{N'} \to C^0(\Omega, \mathbb{R}^n) \) \( x \in W^{1,\infty}(\Omega, \mathbb{R}^n) \) and \( Jx^{N'} \to L^\infty(\Omega, \mathbb{R}^{nm}) \) \( Jx \in L^\infty(\Omega, \mathbb{R}^{nm}) \). Consequently, \( \{ Jx^{N'} \} \) generates a generalized gradient control \( \mu \in \mathcal{G}(K) \).

3) With respect to the topology from (2.2), the set \( \mathcal{Y}(K) \) is sequentially compact.

4) The subset \( \mathcal{G}(K) \subset \mathcal{Y}(K) \) of the generalized gradient controls is sequentially compact as well.

We close this subsection with an extension of Kinderlehrer/Pedregal’s mean value theorem for generalized gradient controls \( \mu \in \mathcal{G}(K) \).

**Theorem 2.8. (Mean value theorem for generalized gradient controls)\(^{31}\)** Assume that \( \Omega \subset \mathbb{R}^m \) is the closure of a strongly Lipschitz domain with \( \partial \in \text{int}(\Omega) \). We consider sequences \( \{ w^N \}, K \) and \( \{ x^N \}, W^{1,\infty}(\Omega, \mathbb{R}^n) \), which satisfy

a) \( w^N \to w \in K \) \( \{ w^N \}, w \in \mathbb{R}^{nm} \) have to be understood as \( (n,m) \)-matrices),

b) \( w^N + Jx^N(t) \in K \) \( \forall t \in \Omega \) \( \forall N \in \mathbb{N} \),

c) \( \{ w^N + Jx^N \} \) generates a generalized gradient control \( \mu \in \mathcal{G}(K) \).

Then there exists a sequence of Lipschitz functions \( \{ \tilde{x}^N \}, W^0_{1,\infty}(\Omega, \mathbb{R}^n) \) with the following properties:

1) \( \lim_{N \to \infty} \| \tilde{x}^N \|_{C^0(\Omega, \mathbb{R}^n)} = 0 \),

2) \( w^N + J\tilde{x}^N(t) \in K \) \( \forall t \in \Omega \) \( \forall N \in \mathbb{N} \),

3) \( \{ w^N + J\tilde{x}^N \} \) generates a constant generalized gradient control \( \nu = \{ \nu \} \in \mathcal{G}(K) \), which may be understood as the average of \( \mu \) with respect to \( t \):

\[
\lim_{N \to \infty} \int_{\Omega} g( w^N + J\tilde{x}^N(t) ) \, dt = \int_{\Omega} \int_{K} g(v) \, d\mu_{t}(v) \, dt
\]

\[
= \lim_{N \to \infty} \int_{\Omega} \int_{K} g( w^N + J\tilde{x}^N(t) ) \, dt = \int_{\Omega} \int_{K} g(v) \, d\nu(t) \, dt \quad \forall g \in C^0(K, \mathbb{R});
\]

4) It holds: \( w = \left( \begin{array}{c} \int_{K} v_{11} \, dv(v) & \cdots & \int_{K} v_{1m} \, dv(v) \\ \vdots & \ddots & \vdots \\ \int_{K} v_{n1} \, dv(v) & \cdots & \int_{K} v_{nm} \, dv(v) \end{array} \right) \).

Theorem 2.8. justifies the definition of an average operator \( A: \mathcal{G}(K) \to \text{rca}^{pr}(K) \), which assigns to any gradient generalized control \( \mu \in \mathcal{G}(K) \) a probability measure \( A(\mu) = \nu \) as its \( t \)-average.

d) Representation theorems for \( f^{(\nu)} \).

For a function \( f \in \mathcal{F}_K \), the author proved that \( f^{(\nu)} \) may be represented in terms of Jacobi matrices in the following way:

\[\text{[Wagner 07d], p. 10, Theorem 2.14., 1].}\]

\[\text{[Berliocchi/Lasry 73], p. 144, Proposition 1, (i); independently proved again in [Kraut/Pickenhain 90], p. 391, Theorem 4.}\]

\[\text{[Wagner 07d], p. 10, Theorem 2.14., 2].}\]

\[\text{Ibid., p. 11, Theorem 2.16., as a generalization of [Kinderlehrer/Pedregal 91], p. 334, Theorem 2.1.}\]

\[\text{\footnote{31}{Ibid., p. 11, Theorem 2.16., as a generalization of [Kinderlehrer/Pedregal 91], p. 334, Theorem 2.1.}}\]
Theorem 2.9. (First representation theorem for \( f^{(q)} \))\(^{32}\) For any function \( f \in \mathcal{F}_K \), its lower semicontinuous quasiconvex envelope \( f^{(q)} : \mathbb{R}^{nm} \to \mathbb{R} \) admits the representation
\[
 f^{(q)}(v_0) = \begin{cases} 
 f^*(v_0) & |v_0| \in \text{int}(K); \\
 \lim_{v \to v_0, v \in R \cap \text{int}(K)} f^*(v) & |v_0| \in \partial K; \\
 (+\infty) & |v_0| \in \mathbb{R}^{nm} \setminus K,
\end{cases}
\]  
(2.6)

where \( R = a_{v_0} \) denotes the ray through \( v_0 \) starting from the origin, and \( f^*(v_0) \) is defined by
\[
 f^*(v_0) = \inf \{ \frac{1}{|\Omega|} \int_{\Omega} f(v_0 + Jx(t)) \, dt \mid x \in W^{1,\infty}(\Omega, \mathbb{R}^n), v_0 + Jx(t) \in K \ (\forall) t \in \Omega \} \in \mathbb{R}. 
\]  
(2.7)

In order to state the second theorem, which shows how to represent \( f^{(q)} \) in terms of probability measures, we define the following subsets:

Definition 2.10. (The set-valued map \( S^{(q)} \))\(^{33}\) For \( v_0 \in K \), we define the following set of probability measures:
\[
 S^{(q)}(v_0) = \{ \nu \in \text{rca}^{pr}(K) \mid \text{there exist sequences } \{ v^N \}, \text{int}(K) \text{ and } \{ x^N \}, W^{1,\infty}(\Omega, \mathbb{R}^n) \text{ with } \]
\[
a) \lim_{N \to \infty} v^N = v_0, \\
b) \lim_{N \to \infty} \| x^N \|_{C^0(\Omega, \mathbb{R}^n)} = 0, \\
c) v^N + Jx^N(t) \in K \ (\forall) t \in \Omega \ \forall N \in \mathbb{N}, \\
d) \{ v^N + Jx^N \} \text{ generates the constant generalized gradient control } \nu = \{ \nu \}.
\]  
(2.8)

Theorem 2.11. (Second representation theorem for \( f^{(q)} \))\(^{34}\) Assume that a function \( f \in \mathcal{F}_K \) is given. Then with the set-valued map \( S^{(q)} : K \to \mathcal{P}(\text{rca}^{pr}(K)) \) from Definition 2.10., it holds for all \( v_0 \in K \) that
\[
 f^{(q)}(v_0) = \text{Min} \left\{ \int_K f(v) \, d\nu(v) \mid \nu \in S^{(q)}(v_0) \right\}. 
\]  
(2.9)

3. Proof of Theorems 1.5. and 1.6.

Proof of Theorem 1.5. · Step 1. From [BALL/KIRCHHEIM/KRISTENSEN 00], we take over the following lemmata.

Lemma 3.1.\(^{35}\) Assume that the closed ball \( C(K(v_0, \delta) \subset \text{int}(K) \), and the function \( \varphi : K(v_0, \delta) \) is separably convex. Denote by \( D \subset K(v_0, \delta) \) the set of the points where the first derivative \( \nabla \varphi(v) \) exists. Then the function \( \nabla \varphi : D \to \mathbb{R} \) is continuous.

Lemma 3.2.\(^{36}\) Assume that the closed ball \( K(v_0, \delta) \subset \text{int}(K) \). Let two functions \( \varphi', \varphi'' : K(v_0, \delta) \to \mathbb{R} \) with \( \varphi'(v_0) = \varphi''(v_0) \) and \( \varphi'(v) \leq \varphi''(v) \) for all \( v \in K(v_0, \delta) \) be given. Assume further that \( \varphi' \) is separately convex, and for \( \varphi'' \) there exists a vector \( a \in \mathbb{R}^{nm} \) with
\[
 \limsup_{w \to a} \frac{1}{|w|} \left( \varphi''(v_0 + w) - \varphi''(v_0) - a^T w \right) \leq 0. 
\]  
(3.1)

\(^{32}\) [WAGNER 07A], p. 29, Theorem 4.1.
\(^{33}\) Synopsis of [WAGNER 07D], p. 15, Definition 3.1. and Lemma 3.2., and p. 21, Theorem 3.9., 2).
\(^{34}\) Ibid., p. 3, Theorem 1.4.
\(^{35}\) [BALL/KIRCHHEIM/KRISTENSEN 00], p. 340, Corollary 2.3.
\(^{36}\) Ibid., p. 341, Corollary 2.5.
Then \( \varphi' \) and \( \varphi'' \) are differentiable in \( v_0 \) with \( \nabla \varphi'(v_0) = \nabla \varphi''(v_0) \).

From Lemma 3.2., we conclude particularly that, for a separately convex function \( \varphi \), the relation (3.1) implies the differentiability in \( v_0 \) (inserting \( \varphi = \varphi' = \varphi'' \)).

- **Step 2.** Let a point \( v_0 \in \text{int} (K) \) be given. In accordance with Theorem 2.11., we choose a probability measure \( \nu_0 \in S^{(q)}(v_0) \) with

\[
f^{(q)}(v_0) = \int f(v) \, d\nu_0(v)
\]

and a sequence \( \{ x^N \} \), \( W^{1,\infty}(\Omega, \mathbb{R}^n) \) such that \( \nu_0 \) is generated by the functions \( v_0 + Jx^N \) in the sense of Definition 2.10. Consequently, we have \( v_0 + Jx^N(t) \in \text{int} (K) \) (\( \forall t \in \Omega \forall N \in \mathbb{N} \)), and the generalized controls \( \delta_{v_0+Jx^N(t)} \) converges in the sense of (2.2) to the constant generalized control \( \{ v_0 \} \). We choose further a number \( 0 < h \leq 1 \) and a point \( w \in \text{int} (K) \). It follows that

\[
v_0 + h (w - v_0) + (1 - h) Jx^N(t) = (1 - h) (v_0 + Jx^N(t)) + h w \in \text{int} (K) \quad (\forall t \in \Omega) \quad (\forall N \in \mathbb{N}).
\]

Then, by Theorem 2.7., 2), a subsequence of the function sequence \( \{ v_0 + h (w - v_0) + (1 - h) Jx^N \} \) generates a generalized gradient control \( \mu \in \mathcal{G}(K) \), whose average \( A(\mu) = \nu_h \in rox^{pr} (K) \) (cf. Theorem 2.8.) belongs to \( S^{(q)}(v_0 + h (w - v_0)) \) (we keep the index \( N \)). Applying Theorems 2.11. and 2.8. again, we obtain

\[
f^{(q)}(v_0 + h (w - v_0)) \leq \int f(v) \, d\nu_h(v) = \lim_{N \to \infty} \frac{1}{|\Omega|} \int_{\Omega} f(v_0 + h (w - v_0) + (1 - h) Jx^N(t)) \, dt.
\]

Together with

\[
f^{(q)}(v_0) = \int f(v) \, d\nu_0(v) = \lim_{N \to \infty} \frac{1}{|\Omega|} \int_{\Omega} f(v_0 + Jx^N(t)) \, dt,
\]

we arrive at the following estimate for the difference quotient of \( f^{(q)}(v) \):

\[
D(w - v_0, h) = \frac{1}{h} \left( f^{(q)}(v_0 + h (w - v_0)) - f^{(q)}(v_0) \right)
\]

\[
\leq \lim_{N \to \infty} \frac{1}{|\Omega|} \int_{\Omega} \frac{1}{h} \left( f(v_0 + Jx^N(t) + h (w - v_0 - Jx^N(t))) - f(v_0 + Jx^N(t)) \right) \, dt.
\]

- **Step 3.** By assumption, the function \( \tilde{f} \) is continuously differentiable on some open neighbourhood of \( K \). Consequently, \( \tilde{f} \) admits a Taylor expansion

\[
\tilde{f}(v + h (w - v)) - \tilde{f}(v) - \nabla \tilde{f}(v)^T h (w - v) = R(v, h (w - v))
\]

for all \( v, w \in K \) where, for fixed \( w \in K \), \( R \) is continuous on \( K \) as a function of \( v \). Moreover, the continuous differentiability of \( \tilde{f} \) implies its Fréchet differentiability, which may be expressed as follows: \( \forall \varepsilon > 0 \exists \delta (\varepsilon) > 0 \) such that for all \( 0 < h \leq 1 \) and all \( v, w \in K \) the implication

\[
| h (w - v) | \leq \delta (\varepsilon) \implies | R(v, h (w - v)) | \leq \varepsilon \cdot | h (w - v) |
\]

holds ([Ioffe/Tichomirov 79], p. 36.). It follows that

\[
\lim_{h \to 0} \frac{R(v, h (w - v))}{h} = 0
\]
for all \( v, w \in K \), and the function sequence

\[
\left\{ \frac{R(v, (1/N)(w-v))}{1/N} \right\},
\]

which is uniformly convergent on \( K \), possesses a continuous majorant. Consequently, we obtain:

\[
D(w - v_0, h) \leq \lim_{N \to \infty} \frac{1}{|\Omega|} \int_{\Omega} \nabla \tilde{f}(v_0 + Jx^N(t))(w - v_0 - Jx^N(t)) dt + \frac{1}{h} \int_{\Omega} R(v_0 + Jx^N(t), h(w - v_0 - Jx^N(t))) dt
\]

\[
= \int_{K} \nabla \tilde{f}(v)^T (w - v) dv_0(v) + \int_{K} \frac{R(v, h(w-v))}{h} dv_0(v).
\]

From the majorized convergence \( \lim_{h \to 0} R(v, h(w-v))/h = 0 \) for all \( v \in K \), it follows that

\[
D^+(w - v_0) = \limsup_{h \to 0+0} D(w - v_0, h) \leq \int_{K} \nabla \tilde{f}(v)^T ((w - v_0) + (v_0 - v)) dv_0(v) = E(w - v_0).
\]

**Step 4.** We invoke two lemmata about quasiconvex functions, which may take the value \((+\infty)\):

**Lemma 3.3.**\(^{37}\) Let \( v_0 \in \mathbb{R}^m \) and \( \mu > 0 \) be given. Together with \( g(v) : \mathbb{R}^m \to \mathbb{R} \), the function \( h(v) = g(v_0 + \mu v) \) is quasiconvex as well.

**Lemma 3.4.**\(^{38}\) Let a convex body \( K \subset \mathbb{R}^m \) and a quasiconvex function \( f : \mathbb{R}^m \to \mathbb{R} \) with \( f \mid K \) be given. Assume that \( f \mid K \) is bounded. Then the restriction \( f \mid \text{int}(K) \) is rank one convex.

By Lemma 3.3., together with \( f^{qc}(v) \), the function \( g(v) = f^{qc}(v_0 + h v) \) is quasiconvex with respect to \( v \). Since \( \text{dom}(g) = \{ v \in \mathbb{R}^m : v \in \frac{1}{\delta} K - \{ \frac{1}{\delta} v_0 \} \} \) and \( v_0 \in \text{int}(K) \), we obtain \( K(\sigma, \delta) \subset \text{int}(\frac{1}{\delta} K - \{ \frac{1}{\delta} v_0 \}) \) for sufficiently small \( \delta > 0 \) and for all sufficiently small \( h > 0 \). Then by Lemma 3.4., the quasiconvexity of \( g(v) \) (in the sense of Definition 1.1.) implies its rank one convexity and separate convexity on \( K(\sigma, \delta) \).

Consequently, for all \( w \in \text{int}(K) \) and all sufficiently small \( h > 0 \), \( D(w - v_0, h) \) is separately convex as a function of \( w - v_0 \) on the interior of its (convex) effective domain, and particularly on \( (w - v_0) \in K(\sigma, \delta) \). In the pointwise forming of the upper limit in (3.13), this property is carried over to \( D^+(w - v_0) \). Moreover, \( D^+ \) is positively homogeneous as a function of \( (w - v_0) \) with \( D^+(v_0 - v_0) = 0 \). \( E(w - v_0) \) is an affine-linear function of \( (w - v_0) \).

**Lemma 3.5.** Consider a positively homogeneous function \( A : \mathbb{R}^m \to \mathbb{R} \) and an affine-linear function \( B : \mathbb{R}^m \to \mathbb{R} \) with \( A(\sigma) = 0 \) and \( A(v) \leq B(v) \) for all \( v \in \mathbb{R}^m \). Then it follows that \( A(v) \leq B(v) - B(\sigma) \) for all \( v \in \mathbb{R}^m \).

**Proof.** Assume on the contrary that there exists some \( z \in \mathbb{R}^m \) with \( B(z) - B(\sigma) < A(z) \). Then for all \( 0 < \gamma \), it follows that

\[
\left( \gamma B(z) + (1 - \gamma) B(\sigma) \right) - \left( (1 - \gamma) B(\sigma) - \gamma B(z) \right) < \gamma A(z) = A(\gamma z) \leq B(\gamma z) \implies \quad (3.14)
\]

\[
0 < A(\gamma z) - \left( B(\gamma z) - B(\sigma) \right) \leq B(\sigma) \implies (3.15)
\]

\[
0 < \gamma \left( A(z) - B(z) + B(\sigma) \right) \leq B(\sigma), \quad (3.16)
\]

\(^{37}\) WAGNER 07a, p. 7, Lemma 2.10., 3).

\(^{38}\) Ibid., p. 7, Theorem 2.12.
and the member \( \gamma \left( A(z) - B(z) + B(\sigma) \right) \) must be bounded. In contradiction to our assumption, this is possible only for \( A(z) - B(z) + B(\sigma) = 0 \). ■

In consequence of Lemma 3.5., the relation \( D^+(w - v_0) \leq E(w - v_0) \) for all \( w \in K(v_0, \delta) \) implies the inequality

\[
D^+(w - v_0) \leq E^+(w - v_0) = E(w - v_0) - E(v_0 - v_0) \tag{3.17}
\]

for all \( w \in K(v_0, \delta) \). Applying now Lemma 3.2. to \( \varphi'(w - v_0) = D^+(w - v_0) \) and \( \varphi'' = E^+(w - v_0) \), we observe that both functions are differentiable in \( (v_0 - v_0) \), and it holds that

\[
\nabla D^+(v_0 - v_0) = \nabla E^+(v_0 - v_0) = \left( \int_K \frac{\partial \tilde{f}}{\partial v_{ij}}(v) \, dv_0(v) \right)_{i,j}. \tag{3.18}
\]

We conclude that the functions \( D^+ \) and \( E^+ \) coincide for all \( w \in K(v_0, \delta) \) and, consequently, for all \( w \in \mathbb{R}^{nm} \). Thus we obtain

\[
D^+(w - v_0) = \sum_{i,j} \int_K \frac{\partial \tilde{f}}{\partial v_{ij}}(v) \, dv_0(v) \left( w_{ij} - v_{0,ij} \right). \tag{3.19}
\]

\textbf{Step 5.} We will apply Lemma 3.2. again in order to confirm the differentiability of \( f^{(q_c)} \) in \( v_0 \) (which is a separately convex function on some convex neighbourhood of \( v_0 \in \text{int}(K) \) ). For this purpose, we claim that the relation

\[
\limsup_{w \to v} \frac{1}{|w|} \left( f^{(q_c)}(v_0 + w) - f^{(q_c)}(v_0) - \nabla D^+(v_0 - v)^T w \right) \leq 0 \tag{3.20}
\]

holds true. Assuming on the contrary that there exist \( \delta > 0 \) and a sequence \( \{ w^N \} \), \( \text{int}(K) \to \sigma \) with

\[
\delta < \frac{1}{|w^N|} \left( f^{(q_c)}(v_0 + w^N) - f^{(q_c)}(v_0) - \nabla D^+(v_0 - v_0)^T w^N \right) \quad \forall N \in \mathbb{N}, \tag{3.21}
\]

we may select a convergent subsequence of \( \{ w^N / |w^N| \} \to w_0 \) (we keep the index \( N \)). Since \( f^{(q_c)} \) is locally Lipschitz continuous on \( \text{int}(K) \), along this subsequence it holds that

\[
\delta < \frac{1}{|w^N|} \left( f^{(q_c)}(v_0 + w^N) - f^{(q_c)}(v_0 + w_0 \mid w^N \mid) - f^{(q_c)}(v_0) \right) - \nabla D^+(v_0 - v_0)^T \frac{w^N}{|w^N|} \tag{3.22}
\]

\[
\leq \frac{L}{|w^N|} \cdot \left( |v_0 + w^N| - (v_0 + w_0 \mid w^N \mid) \right) \tag{3.23}
\]

\[
+ \frac{1}{|w^N|} \left( f^{(q_c)}(v_0 + |w^N \mid v_0) - f^{(q_c)}(v_0) \right) - \nabla D^+(v_0 - v_0)^T \frac{w^N}{|w^N|} \quad \Rightarrow \quad 0 < \delta < \limsup_{N \to \infty} \cdots = D^+(w_0 + v_0 - v_0) - \nabla D^+(v_0 - v_0)^T \left( (w_0 + v_0) - v_0 \right) = 0, \tag{3.24}
\]

and we arrive at a contradiction. Consequently, \( f^{(q_c)} \) is differentiable for all \( v_0 \in \text{int}(K) \), and the continuity of the derivative follows from Lemma 3.1. ■

**Proof of Theorem 1.6.** If \( \partial K = \text{ext}(K) \) then, by Theorem 3.14, 2 and 3), \( f^{(q_c)} \) is continuous on the whole set \( K \). From Theorem 1.5., we conclude that \( f^{(q_c)} \) is continuously differentiable in \( v_0 \in \text{int}(K) \), and it holds that

\[
\frac{\partial f^{(q_c)}}{\partial v_{ij}}(v_0) = \int_K \frac{\partial \tilde{f}}{\partial v_{ij}}(v) \, dv_0(v) \tag{3.25}
\]
where \( \nu_0 \) must be chosen as an element of \( S^{(qc)}(v_0) \) with \( f^{(qc)}(v_0) = \int_K f(v) \, d\nu_0(v) \). In particular, it follows that \( f^{(qc)} \) is Lipschitz continuous on \( \text{int}(K) \) with a constant

\[
C \cdot \sup_{v \in K} |\nabla \tilde{f}(v)|.
\]

(3.26)

Then the Lipschitz condition may be extended to \( \partial K \). ■

References.

20. [Lur’e 75] Lur’e, K. A. (Лурье, К. А.): Оптимальное управление в задачах математической физики. Наука; Москва 1975

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