Limiters and Riemann solvers – the basic ingredients of Finite Volume schemes

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We start with 1d linear advection, end with multidimensional nonlinear systems.
The simplest discretization of linear advection is unconditionally unstable.

\[
\frac{q_i^{n+1} - q_i^n}{\Delta t} - \frac{aq_i^n - aq_{i-1}^n}{2\Delta x} = 0
\]
Stable schemes are obtained with better time- or space discretizations.

Change time difference:

- Leapfrog: \[ \frac{q_{i}^{n+1} - q_{i}^{n-1}}{2\Delta t} \]
- Lax-Friedrichs: \[ \frac{q_{i}^{n+1} - \frac{1}{2}(q_{i+1}^{n} + q_{i-1}^{n})}{\Delta t} \]
- Rusanov/LLF: \[ \frac{q_{i}^{n+1} - \frac{1}{2}(\nu q_{i+1}^{n} + 2(1 - \nu)q_{i}^{n} + \nu q_{i-1}^{n})}{\Delta t} \]

Change space difference:

- Upwind: \[ \frac{aq_{i}^{n} - aq_{i-1}^{n}}{\Delta x} \]
The different forms of the upwind scheme lead to different generalizations.

Difference form

\[ q_i^{n+1} = q_i^n - \nu \Delta q_{i-1/2} \]

Fluctuation form

\[ q_i^{n+1} = q_i^n + \frac{\Delta t}{\Delta x} a (-\Delta w_{i-1/2}^n), \quad w_{i-1/2}^n = \Delta q_{i-1/2}^n \]

Viscous form/Flux form

\[ q_i^{n+1} = q_i^n + \frac{\Delta t}{2\Delta x} \left[ f(q_{i+1}^n) - f(q_{i-1}^n) \right] + \frac{\Delta t}{2\Delta x} a \left[ q_{i+1}^n - 2q_i^n + q_{i-1}^n \right] \]

\[ = q_i^n - \frac{\Delta t}{\Delta x} \left[ F_{i+1/2}^n - F_{i-1/2}^n \right] \]

with \[ F_{i+1/2} = f(q_i) = \frac{1}{2} \left[ f(q_{i+1}^n) + f(q_i^n) \right] - \frac{1}{2} |a| \Delta q_{i+1/2} \]
Lax-Wendroff can be written as a correction of first order upwind.

\[ q_{i}^{n+1} = q_{i}^{n} - \left[ \nu + \frac{1}{2} \nu (1 - \nu) \left( \frac{1}{r_{i}^{n}} - 1 \right) \right] \Delta q_{i-1/2}^{n} \]

with

\[ r = \frac{\Delta q_{up}}{\Delta q_{down}} \]
\[ \nu = \frac{\text{distance covered in } \Delta t}{\Delta x} \]
Beam-Warming can be written as a correction of first order upwind.

$$q_{i}^{n+1} = q_{i}^{n} - \left[ \nu + \frac{1}{2} \nu (1 - \nu) \left( \frac{r_{i}^{n}}{r_{i}^{n}} - r_{i-1}^{n} \right) \right] \Delta q_{i-1/2}^{n}$$

with

$$r = \frac{\Delta q_{up}}{\Delta q_{down}}$$

$$\nu = \frac{\text{distance covered in } \Delta t}{\Delta x}$$
Fromm scheme can be written as a correction of first order upwind.

\[ q_{i}^{n+1} = q_{i}^{n} - \left[ \nu + \frac{1}{2} \nu (1 - \nu) \left( \frac{(1 + r_{i}^{n})/2}{r_{i}^{n}} - (1 + r_{i-1}^{n})/2 \right) \right] \Delta q_{i-1/2}^{n} \]

with

\[ r = \frac{\Delta q_{\text{up}}}{\Delta q_{\text{down}}} \]

\[ \nu = \frac{\text{distance covered in } \Delta t}{\Delta x} \]
TVD-schemes can be written as a correction of first order upwind.

General Case

$q_i^{n+1} = q_i^n - \left[ \nu + \frac{1}{2} \nu (1 - \nu) \left( \frac{\varphi(r_i^n)}{r_i^n} - \varphi(r_{i-1}^n) \right) \right] \Delta q_{i-1/2}^n$

with

$r = \frac{\Delta q_{up}}{\Delta q_{down}}$

$\nu = \frac{\text{distance covered in } \Delta t}{\Delta x}$
A mean function should be at least consistent and homogeneous.

\[ \varphi(r) = \mathcal{M}(r, 1) \]

**Consistency**

\[ \mathcal{M}(a, a) = a \quad \varphi(1) = 1 \]

**Inclusion**

\[ \min\{a, b\} \leq \mathcal{M}(a, b) \leq \max\{a, b\} \quad \min\{r, 1\} \leq \varphi(r) \leq \max\{r, 1\} \]

**Homogeneity**

\[ \mathcal{M}(\lambda a, \lambda b) = \lambda \mathcal{M}(a, b) \]

**Symmetry**

\[ \mathcal{M}(a, b) = \mathcal{M}(b, a) \quad \varphi(1/r) = \varphi(r)/r \]

**Monotonicity**

\[ \mathcal{M}(\cdot, b) \text{, } \mathcal{M}(a, \cdot) \text{ increasing} \quad \varphi(r) \text{, } r \varphi(1/r) \text{ increasing} \]
The TVD-region is much larger than the Sweby region.

\[-\frac{2}{1-|\nu|} \leq \frac{\varphi(r)}{r} - \varphi(R) \leq \frac{2}{|\nu|}\]

\[-2 \leq \frac{\varphi(r)}{r} - \varphi(R) \leq 2\]
Third order schemes are upwind biased.

\[ \varphi_3(r) = \left( 1 - \frac{1+|\nu|}{3} \right) + \frac{1+|\nu|}{3} r \]

\[ \varphi_{LW}(r) = 1 \]

\[ \varphi_{BW}(r) = r \]
Limiters might be constructed by sticking to third order as long as possible.

\[
\phi_{\theta}(r) = \min\left\{ \max\left\{ -\frac{(1 - \theta)}{|\nu|} r, \varphi_3(r) \right\}, \max\left\{ -\frac{(1 - \theta)}{1 - |\nu|} r, \theta \frac{2}{|\nu|} r, \theta \frac{2}{1 - |\nu|} \right\} \right\}
\]
Superbee-type limiters provide a good approximation of the amplitude.

Standard example with 200 cells after 10 full rounds ($t = 20$)
The squaring effect spoils the convergence of Superbee type limiters.
There are more methods to obtain higher order.

PPM

Woodward Colella 1984

generalized to higher orders by Rider, Greenough and Kamm 2005

PHM, PRM, Double logarithmic limiter

Marquina, Xiao and Peng 2004, Artebranndt and Schroll, Čada and Torrilhon

ENO/WENO

Harten, Shu, Osher, . . .

measure oscillation of different stencils
2d transport can be realized by DCU or CTU.
One dimensional schemes can be used in multi-d by dimensional splitting.

Godunov

first order

Strang

preserves second order

Lin, Rood (Monthly Weather Review, 1996)
in addition resembles CTU
Geometric limiting can be done in all geometries.

Structured
directionwise
multidimensional

Unstructured grids

Meshless grids (Sonar et al.)
CFL-independent limiters may be generalized for unstructured grids.

Minmod

Many variants

MC

Barth, Jespersen (1989)

Albada

Tu, Alliabadi (2005)
Multi-d limiters should be at least consistent and homogeneous.

Swartz, 1999:
Scalar methods can be applied to systems by characteristic decomposition.

\[ q_t + Aq_x = 0 \]

\[ A = R \Lambda L \quad w = Lq \]

\[ w_t + \Lambda w_x = 0 \]
The numerical flux for linear advection has many nonlinear counterparts.

Godunov: Numerical flux from exact solution of RP

Flux Splitting:

\[ F_{i+1/2} = f^+(q_i) + f^-(q_{i+1}) \]

Enquist Osher:

\[ f^+(q) = \int_0^q \max\{f'(z), 0\} \, dz \]
\[ f^-(q) = \int_0^q \min\{f'(z), 0\} \, dz \]

\[ F_{i+1/2} = \frac{1}{2} (f(q_{i+1}) + f(q_i)) - \frac{1}{2} \int_{q_i}^{q_{i+1}} |f'(z)| \, dz \]

Linearization (Roe)
For nonlinear equations, numerical fluxes can be computed by Roe-linearization.

\[ f(q_{i+1}) - f(q_i) = \int_0^1 f'(\theta q_{i+1} + (1 - \theta)q_i) \, d\theta \, (q_{i+1} - q_i) \]

More general:

\[ q = q(w), \quad b = \frac{dq}{dw}, \quad f(q) = F(w), \quad c = \frac{dF}{dw} \]

\[ \Delta q_{i+1/2} = \tilde{b}(q_{i+1}, q_i) \Delta w_{i+1/2} \]

\[ \Delta F_{i+1/2} = \tilde{c}(q_{i+1}, q_i) \Delta w_{i+1/2} \]

leads to

\[ \tilde{a}(q_{i+1}, q_i) = \frac{\tilde{c}(q_{i+1}, q_i)}{\tilde{b}(q_{i+1}, q_i)} \]
The scalar concepts can be directly generalized to nonlinear systems.

Godunov

Numerical flux from exact solution of RP

\[ F_{i+1/2} = \frac{1}{2} (f(q_{i+1}) + f(q_i)) - \frac{1}{2} \int_{\Gamma} |f'(z)| \, dz \]

Osher-Solomon

other FVS

Steger Warming, Vijayasundaram, van Leer, AUSM, Eberle etc.

Roe Linearization

parameter vectors
HLL-type solvers are Roe-type solvers and vice versa.

\[ \mathbf{v}_{\text{Roe}} = \frac{1}{2} |\tilde{A}(q_r, q_l)| \]

\[ \mathbf{v}_{\text{HLL}} = \frac{1}{2} \frac{S_R + S_L}{S_R - S_L} \tilde{A}(q_r, q_l) - \frac{S_R S_L}{S_R - S_L} I \]

Rusanov/LLF

\[ S_R = -S_L = |\lambda_{\text{max}}| \quad \text{fastest wave speed in RP} \]

Lax-Friedrichs

\[ S_R = -S_L = \frac{\Delta x}{\Delta t} \quad \text{maximal allowed wave speed on grid} \]
Smooth limiters are a good choice for nonlinear waves.

Detail of Shu-Osher problem, $t = 1.8$, 400 cells
Characteristic CFL-dependent limiting enhances the quality of TVD-schemes.

Detail of Toro’s test case 3
Godunov scheme is not always the best choice.
Standard carbuncle fix is based on nonlocal data.
The carbuncle can be addressed from within the Riemann solver.

By entropy consistency
Roe 2008
Bouchut 2003/2004

By adjusting viscosity on shear waves
Kemm 2008
Nishikawa and Kitamura 2008
JST-schemes choose viscosity terms by indicator functions.

differentiable

mainly for steady states and implicit time discretizations

stabilized by implicit time schemes

naive use yields bad results

viscosity via second and fourth derivatives
New schemes provide multidimensional flux calculations.

FVEG

Morton, Warnecke, Lukačova, . . .

2d HLL/HLLC/LFC

Wendroff 1999

Nonstaggered central schemes

Tadmor et al.
A careful choice of limiters and Riemann solvers yields fine results.

Detail of Shu-Osher problem, $t = 1.8$, 400 cells