Notes on a class of flux splitting schemes for the Euler equations

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There is no unique splitting of the Euler flux into an advective and a pressure flux

H-Cusp
\[ \mathbf{u} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ H \end{pmatrix} + \begin{pmatrix} 0 \\ \rho \\ 0 \\ 0 \end{pmatrix} \]

E-Cusp
\[ \mathbf{u} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} + \begin{pmatrix} 0 \\ \rho \\ 0 \\ \rho u \end{pmatrix} \]

VCT-Cusp
\[ \mathbf{u} \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \frac{1}{2} \rho u^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \rho \\ 0 \\ \frac{\gamma}{\gamma-1} \rho u \end{pmatrix} \]

AUSM etc. Zha-Bilgen, AUFS etc. Vázquez Cendón and Toro
There is no splitting of the Euler flux into advection and acoustics

Eigenvalues of split fluxes:

<table>
<thead>
<tr>
<th>Flux Type</th>
<th>Advection Part</th>
<th>Pressure Part</th>
</tr>
</thead>
<tbody>
<tr>
<td>H-cusp</td>
<td>$\gamma u$, $u$ deficient</td>
<td>$0, 0, -(\gamma - 1)u$</td>
</tr>
<tr>
<td>E-cusp</td>
<td>$u$, $u$ deficient</td>
<td>$0, \pm \sqrt{\frac{\gamma - 1}{\gamma}} c$</td>
</tr>
<tr>
<td>VCT-cusp</td>
<td>$0$, $u$ deficient</td>
<td>$0, \frac{u}{2} \pm \sqrt{\frac{u^2}{4} + \frac{\gamma p}{\rho}}$</td>
</tr>
</tbody>
</table>
The splitting can be used directly or with additional upwinding for the pressure part

**General**

\[ F = u \mathbf{U} + P \]

**Direct application**

\[ F_{\text{num}} = \tilde{u} \mathbf{U}_{\text{up}} + P_{\text{num}} \]

**With additional upwinding (e.g. AUFS)**

\[ F_{\text{num}} = \tilde{u} \mathbf{U}_{\text{up}} + M P_{\text{up}} + (1 - M) P_{\text{num}} \]

How to obtain the numerical viscosity for \( P_{\text{num}} \)?
One of the numerical viscosities provided by Sun and Takayama for AUFS vanishes for entropy waves

Steger-Warming

\[
\frac{1}{2c_l} \begin{pmatrix}
    p_l \\
    u_l p_l \\
    v_l p_l \\
    H_l p_l \\
\end{pmatrix} - \frac{1}{2c_r} \begin{pmatrix}
    p_r \\
    u_r p_r \\
    v_r p_r \\
    H_r p_r \\
\end{pmatrix}
\]

AUFS

\[
\frac{1}{2\tilde{c}} \begin{pmatrix}
    p_l - p_r \\
    p_l u_l - p_r u_r \\
    p_l v_l - p_r v_r \\
    \frac{\tilde{c}^2}{\gamma - 1} (p_l - p_r) + \frac{1}{2} (p_l u_l^2 - p_r u_r^2) \\
\end{pmatrix}
\]

Rusanov

\[
\frac{\tilde{c}}{2} \begin{pmatrix}
    p_l - p_r \\
    p_l u_l - p_r u_r \\
    p_l v_l - p_r v_r \\
    E_l - E_r \\
\end{pmatrix}
\]
Exact resolution of entropy waves leads to loss of positivity

Pressure in left half of steady shock test
The AUFS-approach with the different viscosities can be directly transferred from E-cusp to VCT-cusp

Compute viscosity as for original AUFS

Multiply viscosity with

\[
\max \left\{ \frac{u_l}{2} \pm \sqrt{\frac{u_l^2}{4} + \frac{\gamma p_l}{\rho_l}}, \frac{u_r}{2} \pm \sqrt{\frac{u_r^2}{4} + \frac{\gamma p_r}{\rho_r}} \right\} \bigg/ \bar{C}
\]

In last flux component replace \( p \) by \( \frac{\gamma}{\gamma-1} p \)

In advective flux replace total energy by total kinetic energy
Using VCT-cusp stabilizes the original AUFS-scheme

Pressure in left half of steady shock test
After changing the viscosity or transition to VCT, the resolution of entropy waves is still reasonable.

Sod shock-tube with different AUFS-type schemes.
Due to the viscosity on the shear wave AUFS-schemes don’t show the carbuncle but overshoots near strong shocks.

Colliding flow \((t = 50)\) with different AUFS-type schemes.
The viscosity on the shear wave may be reduced by a reverse carbuncle fix

Simple fix

Multiply 3rd component of viscosity by

\[
\left( \frac{|p_l - p_r|}{p_l + p_r} \right)^\alpha \cdot \left( \frac{|u_l - u_r|}{|u_l| + |u_r|} \right)^\beta \quad \text{with} \quad \alpha, \beta \in [0, 1]
\]

Elaborate fix

Compute norm \( \|r\| \) of residual of RH-condition for contact
Apply monotone increasing function \( g : [0, \infty) \rightarrow [0, 1] \)
Multiply 3rd component of viscosity by \( g(\|r\|) \)
For $\alpha = 1/4$, $\beta = 0$ and VCT with Rusanov-type viscosity, no carbuncle is found in most tests . . .

Colliding flow ($t = 30$) and steady shock with shear correction
...but not all

Quirk test without and with shear correction
The full potential, especially of VCT, is not yet exploited

Direct application of splitting

ongoing work of Vázquez Cendón and Toro

different standard Riemann solvers for pressure part

for E-cusp already many failed schemes

With additional upwinding

for VCT take into account the asymmetry in wave speeds