CFL-Number-dependent TVD-Limiters

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TVD-schemes can be written as a correction of first order upwind.

General Case

\[ q_{i}^{n+1} = q_{i}^{n} - \left( \nu + \frac{1}{2} \nu(1 - \nu) \left( \frac{\varphi(r_i^n)}{r_i^n} - \varphi(r_{i-1}^n) \right) \right) \Delta q_{i-1/2}^n \]

with

\[ r = \frac{\Delta q_{up}}{\Delta q_{down}} \]

\[ \nu = \frac{\text{distance covered in } \Delta t}{\Delta x} \]
The TVD-region is much larger than the Sweby region

\[-2 \leq \frac{\varphi(r)}{r} - \varphi(R) \leq 2\]
First order does not imply high diffusion

Standard test with CFL-number 0.5
Third order schemes are upwind biased

\[
\varphi_3(r) = \left(1 - \frac{1 + |\nu|}{3}\right) + \frac{1 + |\nu|}{3} r \\
\varphi_{LW}(r) = 1 \\
\varphi_{BW}(r) = r
\]
Limiters might be constructed by sticking to third order as long as possible.

\[ \varphi_\theta(r) = \min \left\{ \max \left\{ \left( 1 - \theta \right) \frac{2}{|\nu|}, \varphi_3(r) \right\}, \max \left\{ \frac{2}{1 - |\nu|} r, \frac{2}{|\nu|} \theta r, \theta \frac{2}{1 - |\nu|} \right\} \right\} \]
Third order does not guarantee for good representation of the amplitude

Standard example with 200 cells after 10 full rounds ($t = 20$)
Limiters might employ as much compression as possible without loosing inclusion
Superbee-type limiters provide a good approximation of the amplitude

Standard example with 200 cells after 10 full rounds ($t = 20$)
Extreme limiters satisfy some optimality conditions

Upper bound

1st order

optimal for piecewise constant data

Després, Lagoutière (2001)

CFL-Superbee

2nd order

optimal for piecewise linear data

Bokanowski, Zidani (2007)

For piecewise smooth data?
The β-Limiter is a compromise between 3rd order and CFL-Superbee

\[ \phi_\beta(r) = \max\left\{ 0, \min\{\phi_{UB}(r), \quad \right. \]

\[ \max\{1 + (\phi'_3 - \beta/2)(r - 1), 1 + (\phi'_3 + \beta/2)(r - 1)\} \quad \left. \right\} \]

fixed: \( \beta = \frac{2}{3} \)  
by Harten Switch: \( \beta = \frac{1}{3} + \frac{2}{3} \cdot \frac{|1 - r|}{1 + |r|} \)
In most cases fixed $\beta = 2/3$ is the best choice.
Superpower generalizes the Power limiters by Serna and Marquina

\[ \phi_{sp}(r) = \max\{0, \phi_3(r) \left(1 - \left|\frac{1 - |r|}{1 + |r|}\right|^{p(r)}\right)\} \]

\[ p(r) = \begin{cases} 
\frac{2}{|\nu|} \cdot 2(1 - \phi_3'), & r \leq 1 \\
\frac{2}{|1-\nu|} \cdot 2\phi_3', & r \geq 1
\end{cases} \]
For nonlinear equations, the left bound of the TVD-region changes

Jeng and Payne (1995):

\[
0 \leq \frac{\varphi(r)}{r} \leq \frac{2}{|\nu|} \frac{1 - |\nu_{up}|}{1 - |\nu|}
\]

\[
0 \leq \varphi(r) \leq \frac{2}{1 - |\nu|}
\]

If \( \varphi(r) \equiv 0 \) for \( r \leq 0 \) then

\[
0 \leq \varphi(r) \leq \min\left\{ \frac{2}{|\nu|} \frac{1 - |\nu_{up}|}{1 - |\nu|} r, \frac{2}{1 - |\nu|} \right\}
\]
The nonlinear TVD-condition prevents overshoots

Detail of Burgers with sinusoidal initial data
Smooth limiters are a good choice for nonlinear waves

Detail of Shu-Osher problem, $t = 1.8$, 400 cells
There are still some open questions

TVD-schemes on non equidistant grids?

normalized variables (Leonard, Waterson)

CFL-independent limiters

TVD-condition in the presence of source terms?

no studies known

grateful for all hints

TVD-condition in 2d or 3d?

first answers by Kim et al

CFL-independent limiters
Exploiting the full potential of TVD-schemes pays off

Detail of Shu-Osher problem, $t = 1.8$, 400 cells